

# TAME STRUCTURES VIA MULTIPLICATIVE CHARACTER SUMS ON VARIETIES OVER FINITE FIELDS

MINH CHIEU TRAN

**ABSTRACT.** We study the model theory of  $(\mathbb{F}; <_\chi)$  where the field  $\mathbb{F}$  is an algebraic closure of a finite field and  $<_\chi$  is an ordering on the multiplicative group  $\mathbb{F}^\times$  induced by a group embedding  $\chi: \mathbb{F}^\times \rightarrow \mathbb{C}^\times$ . Using number-theoretic bounds on multiplicative character sums over finite fields and Weyl's criterion for equidistribution, we establish a number of properties about the interaction between  $<_\chi$  and the underlying field structure. We obtain a first-order axiomatization of these properties and show that the resulting theory is strongly model complete and has NTP<sub>2</sub>.

## 1. INTRODUCTION

Pseudo-finite fields are important examples of tame structures in model theory; see [Cha97] for a survey. The study of these structures began with Ax, who used results about counting points on varieties over finite fields and Chebotarev's density theorem to show that a field is pseudo finite if and only if it is elementarily equivalent to a non-principal ultra product of finite fields [Ax68]. In this paper we show that related results about multiplicative character sums on varieties over finite fields yield tame structures in a rather different fashion. This answers a version of a question of van den Dries, Hrushovski and Kowalski which we loosely interpret as asking for applications of character and exponential sums in model theory. (However, we do not use results in [Kow07] as they suggested).

Throughout,  $\mathbb{F}$  is an algebraic closure of a finite field and  $\chi$  is a group embedding from  $\mathbb{F}^\times$  to  $\mathbb{C}^\times$ , where  $\mathbb{F}^\times$  and  $\mathbb{C}^\times$  are the multiplicative groups of  $\mathbb{F}$  and the field of complex number  $\mathbb{C}$  respectively. Let  $\mathbb{U}_{(p)} \subseteq \mathbb{C}^\times$  be the group of roots of unity with order coprime to  $p$  when  $p$  is prime and the group of roots of unity when  $p$  is zero. Let  $\mathbb{T} \subseteq \mathbb{C}^\times$  be the unit circle. Then  $\text{Im}\chi = \mathbb{U}_{(p)} \subseteq \mathbb{T}$  where  $p = \text{char}(\mathbb{F})$ . We denote by  $<$  the natural ordering on the field of real numbers  $\mathbb{R}$ . Identifying the interval  $[0, 1) \subseteq \mathbb{R}$  with  $\mathbb{T}$  via  $\alpha \mapsto e^{2\pi i \alpha}$ , the above  $<$  induces cyclic orderings on  $\mathbb{T}$  and  $\mathbb{U}_{(p)}$  for  $p$  either prime or zero which we also denote by  $<$ . Define  $<_\chi$  on  $\mathbb{F}^\times$  to be the pullback of  $<$  on  $\mathbb{T}$  by  $\chi$  and view  $<_\chi$  as a binary relation on  $\mathbb{F}$ . We will show that  $(\mathbb{F}; <_\chi)$  is model theoretically tame for all  $\mathbb{F}$  and  $\chi$  as above.

We can think of the above  $(\mathbb{F}; <_\chi)$  as an amalgam of two simpler structures: the algebraically closed field  $\mathbb{F}$  and with the cyclically ordered group  $(\mathbb{F}^\times; <_\chi)$ . The latter can be identified via  $\chi$  with  $(\mathbb{U}_{(p)}; <)$  where  $p = \text{char}(\mathbb{F})$ . This suggests studying the model theory of  $(\mathbb{F}; <_\chi)$  by first analyzing each of these two structures and then understanding the way they are “glued” together.

---

*Date:* April 13, 2017.

2010 *Mathematics Subject Classification.* Primary 03C65; Secondary 03B25, 03C10, 03C64, 11T24, 12L12.

The above  $(\mathbb{F}; <_\chi)$  is more suitably studied in a slightly richer language which does not introduce extra definable sets. This is necessary as  $(\mathbb{U}_{(p)}; <)$  does not admit quantifier elimination in the language of groups with a relation symbol for  $<$  when  $p$  is prime. For  $c \in \mathbb{U}_{(p)}$  with  $p$  either prime or zero and  $n \in \mathbb{N}^{\geq 1}$ , define the “winding number”  $\text{wn}(c, n)$  as the number of elements of the set

$$\{k \in \mathbb{Z} : 0 \leq k \leq n-1, c^{k+1} < c^k\}.$$

For  $p$  either prime or zero, let  $\mathcal{P}$  denote the family  $(\mathcal{P}_n^r)_{n,r}$  of unary relations on  $\mathbb{U}_{(p)}$  where  $n$  ranges over  $\mathbb{N}^{\geq 1}$ ,  $r$  is in  $\{0, \dots, n-1\}$  and  $\mathcal{P}_n^r \subseteq \mathbb{U}_{(p)}$  is the set

$$\{a \in \mathbb{U}_{(p)} : \text{there is } c \in \mathbb{U}_{(p)} \text{ with } c^n = a \text{ and } \text{wn}(c, n) = r\}.$$

The expansion  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  of  $(\mathbb{U}_{(p)}; <)$  by the family  $\mathcal{P}$  is then a structure in the language  $L_m$  extending the language of groups with a binary predicate symbol for  $<$  and a family of unary predicate symbols for  $\mathcal{P}$ . With  $(\mathbb{F}^\times; <_\chi)$  identified with  $(\mathbb{U}_{(p)}; <)$  via  $\chi$  where  $p = \text{char}(\mathbb{F})$ , define  $\mathcal{P}_\chi$  on  $\mathbb{F}^\times$  to be the pullback of  $\mathcal{P}$  by  $\chi$  and view  $\mathcal{P}_\chi$  as a family of unary relations on  $\mathbb{F}$ . Then  $(\mathbb{F}^\times; <_\chi, \mathcal{P}_\chi)$  is an  $L_m$ -structure isomorphic to  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  where  $p = \text{char}(\mathbb{F})$  and  $(\mathbb{F}; <_\chi, \mathcal{P}_\chi)$  is a structure in the language  $L_c$  obtained by combining  $L_m$  and the language of rings  $L_r$ . We call the structures  $(\mathbb{F}; <_\chi, \mathcal{P}_\chi)$  for varying  $\mathbb{F}$  and  $\chi$  the **standard models**.

We isolate the first-order properties of the standard models. Observe that all standard models belong to the class of  $L_c$  structures  $(F; <, \mathcal{P})$  where  $<$  is a subset of  $F^\times \times F^\times$ ,  $\mathcal{P}$  is a family of subsets of  $F^\times$ , and the following conditions are satisfied for some  $p$  either prime or zero:

- (1)  $F$  is an algebraically closed field of characteristic  $p$ ;
- (2)  $(F^\times; <, \mathcal{P}) \models T_{m,p}$  where  $T_{m,p}$  is the theory of  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  in  $L_m$ .

We call the class of  $L_c$ -structure specified above  $\text{ACFO}^-$ . This goes against the usual convention of reserving such names for a theory in a certain language. To recover internal consistency, we define a theory  $T$  in a language  $L$  as a class of  $L$ -structures which is first-order axiomatizable in  $L$  (instead of a set of  $L$ -sentences as usual) and modify the standard concepts and notions of model theory in obvious ways to suit this setting. The class  $\text{ACFO}^-$  is not obviously first-order axiomatizable but will eventually be shown to be so. Before that happens we will still abuse terminology and call a structure in  $\text{ACFO}^-$  a model of  $\text{ACFO}^-$ . Several similar situations are encountered in this paper and we deal with them in the similar fashion. Replacing  $T_{m,p}$  in (2) with the class  $T_{m,p}(\forall)$  of its  $L_c$ -substructures, we obtain the definition of  $\text{ACFO}^=$ , where the superscript “=” is read as “double minus”. For  $p$  either prime or zero, the theories  $\text{ACFO}_p^-$  and  $\text{ACFO}_p^=$  are obtained from  $\text{ACFO}$  and  $\text{ACFO}^=$  by fixing  $p$  respectively.

Heuristically, the models of  $\text{ACFO}^-$  are obtained by “gluing” a model of ACF and a model of  $T_m = \bigcup_p T_{m,p}$  in such a way that the multiplicative group of the model of ACF matches the underlying multiplicative group of the model of  $T_m$ . Our guiding intuition is the following: in a standard model  $(\mathbb{F}; <_\chi, \mathcal{P}_\chi)$ , the components  $\mathbb{F}$  and  $(\mathbb{F}^\times; <_\chi, \mathcal{P}_\chi)$  interact with one another in a “random” or “generic” manner on top of their obvious agreement on  $\mathbb{F}^\times$ . This is analogous to known examples of adding a generic predicate as in [CP98] and [Che14], amalgamating simple structures as in [Tsu01] and adding a generic linear order as in [SS12]. We will adapt the techniques in these papers to establish the tameness of our structure.

We make the above precise. Suppose  $(F; <, \mathcal{P}_n^r) \models \text{ACFO}^-$ . A *quasi-affine variety* (over  $F$ ) is for us a nonempty open subset of an irreducible closed subset of some  $F^m$ , the latter equipped with its Zariski topology. A quasi-affine variety  $V \subseteq F^m$  is **multiplicatively large** if for all  $k_1, \dots, k_m \in \mathbb{Z}$  not all zero and all  $c \in F^\times$ ,  $V \cap (F^\times)^m$  is not contained in the solution set of the equation

$$x_1^{k_1} \cdots x_m^{k_m} = c.$$

The *order topology* on  $(F^\times)^m$  is defined for  $m = 1$  as the topology which has a basis consisting of semi-open intervals  $\{a : 1 \leq a < c'\}$  and open intervals  $\{a : c < a < c'\}$  with  $c, c' \in F^\times$ , and for  $m > 1$  as the product of the order topologies on the  $m$  copies of  $F^\times$ . We say that  $X \subseteq F^m$  is **order-dense** if  $X \cap (F^\times)^m$  is dense in  $(F^\times)^m$  with respect to the order topology.

We say that a model  $(F; <, \mathcal{P})$  of  $\text{ACFO}^-$  is **generic** if all multiplicatively large quasi-affine varieties over  $F$  are order-dense. Our notion of *generic* has the same underlying idea as notions with the same name in [CP98], [Tsu01] and [SS12] but is closer to the number-theoretic phenomena that we need. Let  $\text{ACFO}$  and  $\text{ACFO}_p$  be the classes of generic models of  $\text{ACFO}^-$  and  $\text{ACFO}_p^-$  respectively. In section 2, we confirm our intuition about the standard models:

**Theorem 1.1.** *The standard models are generic and therefore models of ACFO.*

Our strategy is to prove for a multiplicatively large quasi-affine variety  $V \subseteq \mathbb{F}^m$  in  $(\mathbb{F}; <_\chi, \mathcal{P}_\chi)$  the stronger statement that the image of the set  $V^\times(\mathbb{F}_{q^k}) = V \cap (\mathbb{F}_{q^k}^\times)^m$  under  $\chi$  becomes equidistributed in  $\mathbb{T}$  as  $k \rightarrow \infty$ . This uses number theoretic bounds on character sums and Weyl's criterion for equidistribution.

Section 3 justifies our terminology abuses and gives us compactness:

**Theorem 1.2.** *The classes  $\text{ACFO}^=$ ,  $\text{ACFO}^-$  and  $\text{ACFO}$  are  $\forall\exists$ -axiomatizable.*

We need to show that (2) in the definition of  $\text{ACFO}^-$  is  $\forall\exists$ -axiomatizable. This follows essentially from the quantifier elimination for  $(\mathbb{U}_{(p)}; <, \mathcal{P})$ . Using an idea implicit in [Gün08],  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  can be linked to the structure  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1)$  where  $\mathcal{D} = (\mathcal{D}_n)_{n \in \mathbb{N}_{\geq 1}}$  and  $\mathcal{D}_n \subseteq \mathbb{Z}_{(p)}$  is the predicate for divisibility by  $n$ . By results in [Wei81],  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, 1)$  has quantifier elimination. From this, we can deduce the quantifier elimination of  $(\mathbb{U}_{(p)}; <, \mathcal{P})$ . We also need to show that genericity is  $\forall\exists$ -axiomatizable. This can be reduced to showing that multiplicative largeness is definable in a family. The reduction step has an analogue in [CP98], [Tsu01] and [SS12], but the next step of proving the resulted statement requires new ideas. In particular, our proof uses the Zilber's indecomposability theorem and the fact that every connected algebraic subgroup of an algebraic torus must be a subtorus.

In Section 4 we study the logical tameness  $\text{ACFO}$ . The main theorem is:

**Theorem 1.3.**  *$\text{ACFO}$  is the model completion of  $\text{ACFO}^-$ . Definable sets in an  $\text{ACFO}$ -model are one-to-one coordinate projections of quantifier-free definable sets.*

Given  $(F; <, \mathcal{P}_n^r) \models \text{ACFO}$ , let  $\text{Abs}(F)$  be the prime model of  $\text{ACF}$  contained in  $F$  and let  $<, \mathcal{P}$  be defined on  $\text{Abs}(F)$  by restriction. We deduce a criterion for two models of  $\text{ACFO}$  to be elementarily equivalent:

**Corollary 1.4.** *The  $\text{ACFO}$ -models  $(F; <, \mathcal{P})$  and  $(F'; <, \mathcal{P})$  are elementarily equivalent if and only if  $(\text{Abs}(F); <, \mathcal{P})$  and  $(\text{Abs}(F'); <, \mathcal{P})$  are isomorphic.*

When  $p$  is prime, we obtain a detailed study of  $(\text{Abs}(F); <, \mathcal{P})$  in a model  $(F; <, \mathcal{P})$  of  $\text{ACFO}_p$ . This yields in particular the following converse of Theorem 1.1:

**Proposition 1.5.** *If  $(F; <, \mathcal{P}) \models \text{ACFO}_p$  for  $p$  prime, then  $(\text{Abs}(F); <, \mathcal{P})$  is a standard model and is therefore a model of  $\text{ACFO}_p$ .*

The above is surprising as the given definition and the proof of Theorem 1.1 seem to suggest that the notion of *genericity* is rather weak. Combining with Theorem 1.1 and Theorem 1.3, we get the following analogue of Ax’s theorem:

**Corollary 1.6.** *An  $L_c$ -structure is a model of  $\text{ACFO}_p$  with  $p$  prime if and only if it is elementarily equivalent to a standard model.*

Using Theorem 1.3 and results from computational number theory we obtain:

**Proposition 1.7.** *The set of  $L_c$ -statements which hold in all  $\text{ACFO}$ -models is recursive.*

Let  $\text{acl}_r$  be the algebraic closure operator with respect to  $L_r$  and let  $\text{acl}_c$  and  $\text{dcl}_c$  be the algebraic closure and definable closure operators with respect to  $L_c$ . We get:

**Proposition 1.8.** *In a model of  $\text{ACFO}$ ,  $\text{acl}_c$ ,  $\text{dcl}_c$  and  $\text{acl}_r$  coincide.*

There are a number of new ideas in the proof of the main theorem compared to its counterparts in [CP98], [Tsu01] and [SS12]. First, as mentioned before, our notion of *genericity* is not trivially equivalent to the translation of the notions with the same name in those papers. We therefore need to bridge this gap in the proof that  $\text{ACFO}$  is model complete. In particular, we need to understand the appropriate notion of dimension in  $(\mathbb{U}_{(p)}; <, \mathcal{P})$ . This is done by again linking  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  to  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1)$  and using the results in [Tow13]. Second, the structures in [CP98], [Tsu01] and [SS12] can be seen as free amalgams of two simpler structures, while in  $(\mathbb{F}; <_\chi, \mathcal{P}_\chi)$ ,  $\mathbb{F}$  and  $(\mathbb{F}^\times; <_\chi, \mathcal{P}_\chi)$  agree on  $\mathbb{F}^\times$ . This brings unexpected difficulties. To resolve these, we need among other things the fact that the common reduct of  $\text{ACF}_p$  and  $T_{m,p}$  to the language  $L_g$  of groups has quantifier elimination.

Section 5 studies the combinatorial tameness of  $\text{ACFO}$ . The main result is:

**Theorem 1.9.** *The theory  $\text{ACFO}$  has  $\text{NTP}_2$ .*

We will give a definition of  $\text{NTP}_2$  in section 5 and only provide here a heuristic. A structure is  $\text{NTP}_2$  if it exhibits “algebraic behavior”, “random behavior” and “ordered structure behavior”, but nothing more complicated. In a standard model  $(\mathbb{F}; <_\chi, \mathcal{P}_\chi)$ ,  $\mathbb{F}$  is “algebraic”,  $(\mathbb{F}^\times; <_\chi, \mathcal{P}_\chi)$  is “ordered and algebraic” and the interaction is “random” but there is nothing else, so  $(\mathbb{F}; <_\chi, \mathcal{P}_\chi)$  should have  $\text{NTP}_2$ . The actual proof follows the same strategy as in [Che14]. On the other hand, every model of  $\text{ACFO}$  is not simply interpreting a dense linear ordering. In addition:

**Proposition 1.10.** *Every model of  $\text{ACFO}$  interprets a random graph and hence has IP.*

The latter is also a consequence of a result in [SS12], which tells us more generally that adding any linear ordering to  $\mathbb{F}$  gives us a structure with IP.

#### ACKNOWLEDGEMENTS

I am grateful to many individuals for their help throughout this work. My advisor Lou van den Dries brought my attention to results on character sums and provided me with a lot of support and guidance. Remarks and suggestions by Erik Walsberg were extremely helpful. The proof of Lemma 2.2 in section 2 was explained to me by Chee Whye Chin. Dane Skabelund pointed to me many good references. Finally, William Balderrama and many others helped correct many grammar mistakes.

## 2. GENERICITY OF THE STANDARD MODELS

Throughout  $k$  and  $l$  range over the integers,  $m$  and  $n$  range over the natural numbers (which include zero) and  $p$  ranges over the set  $\{n \in \mathbb{N} : n \text{ is zero or prime}\}$ . Let  $x = (x_1, \dots, x_m)$  and  $y = (y_1, \dots, y_n)$  be tuples of variables. If  $a$  is in  $X^m$  then  $a = (a_1, \dots, a_m)$  with  $a_i \in X$  for  $i \in \{1, \dots, m\}$ .

Assume also in this section that  $k \geq 1$ ,  $p = \text{char}(\mathbb{F})$ ,  $q = p^l$  for  $l \geq 1$ ,  $\mathbb{F}_q$  is the subfield of  $\mathbb{F}$  with  $q$  elements,  $P$  is in  $\mathbb{F}_q[x]$  and  $V \subseteq \mathbb{F}^m$  is a quasi-affine variety of dimension  $d$  definable in the field sense over  $\mathbb{F}_q$ . Toward Theorem 1.1, we need two number theoretic results:

**Lemma 2.1** (Lang-Weil Estimate).  $|V(\mathbb{F}_{q^k})| = q^{kd} + O(q^{k(d-\frac{1}{2})})$  as  $k \rightarrow \infty$ .

*Proof.* This is a weaker form of Theorem 1 in [LW54].  $\square$

**Lemma 2.2.** *If  $P$  is not constant on  $V$ , then*

$$\left| \sum_{a \in V(\mathbb{F}_{q^k})} \chi(P(a)) \right| = O(q^{k(d-\frac{1}{2})}) \text{ as } k \rightarrow \infty.$$

*Proof.* The result is well known and follows from Deligne's proof of the generalization of Riemann hypothesis for function fields [Del80]. More particularly, we can apply Remark 1.18 in [Del77] to the pullback of the associated Kummer sheaf  $\mathcal{L}_\chi$  to  $V$  by  $P$  (see 1.7 of [Del77]). In the appendix we provide a more elementary proof depending only on a Weil style bound.  $\square$

We will also need a variation of Weyl's criterion for equidistribution. For  $b, b' \in \mathbb{T}^m$ , we write  $b < b'$  if  $b_i < b'_i$  for all  $i \in \{1, \dots, m\}$ . For  $b, b' \in \mathbb{T}^m$  such that  $b < b'$ , set

$$V(b, b') = \prod_{j=1}^m (1(b'_j) - 1(b_j)) \text{ with } 1: \mathbb{T} \rightarrow [0, 1] \subseteq \mathbb{R} \text{ mapping } e^{2\pi i \alpha} \text{ to } \alpha.$$

For the rest of the section,  $(X_k)_{k \in \mathbb{N}}$  is a sequence of finite subsets of  $\mathbb{T}^m$ . We say that  $(X_k)_{k \in \mathbb{N}}$  becomes *equidistributed* in  $\mathbb{T}^m$  if

$$\lim_{k \rightarrow \infty} \left( \frac{1}{|X_k|} |\{a \in X_k : b < a < b'\}| \right) = V(b, b') \text{ for all } b, b' \in \mathbb{T}^m \text{ with } b < b'.$$

**Lemma 2.3** (Weyl's Criterion). *The sequence  $(X_k)_{k \in \mathbb{N}}$  becomes equidistributed in  $\mathbb{T}^m$  if and only if*

$$\lim_{k \rightarrow \infty} \left( \frac{1}{|X_k|} \sum_{a \in X_k} a_1^{l_1} \dots a_m^{l_m} \right) = 0 \text{ for all } l \in \mathbb{Z}^m \setminus \{(0, \dots, 0)\}.$$

*Proof.* The proof is the same as that for Weyl's criterion for equidistribution of sequence. See for example page 112 of [SS03].  $\square$

*Proof of Theorem 1.1.* It suffices to show that if  $V \subseteq (\mathbb{F}^\times)^m$  is multiplicatively large and  $X_k$  is the image of  $V(\mathbb{F}_{q^k})$  under  $\chi$ , then the sequence  $(X_k)_{k \in \mathbb{N}}$  becomes equidistributed. Using Weyl's criterion, we need to verify that

$$\lim_{k \rightarrow \infty} \left( \frac{1}{|V(\mathbb{F}_{q^k})|} \sum_{a \in V(\mathbb{F}_{q^k})} \chi(a_1^{l_1} \dots a_m^{l_m}) \right) = 0.$$

Apply Lemma 2.1 and Lemma 2.2 with  $P = x_1^{l_1} \dots x_m^{l_m}$  noting that  $P$  is non-constant on  $V$  as  $V$  is multiplicatively large.  $\square$

## 3. AXIOMATIZATION

In this section, we use the following conventions in addition to those introduced in the first paragraph of the preceding section. Let  $L$  be a language. Denote by  $[L]$  the class of all  $L$ -structures. If  $T$  is a class of  $L$ -structures, define  $(T, \hookrightarrow)$  as the category whose objects are  $T$ -models and whose morphisms are  $L$ -embeddings. Suppose  $(M; \dots) \preceq (M'; \dots)$  are  $L$ -structures and  $X \subseteq M^m$  is definable. Then we set  $X(M')$  to be the subset of  $(M')^m$  defined by any  $L$ -formula with parameters over  $M$  that defines  $X$ . Suppose  $R$  is a relation on a set  $M$  and  $M' \subseteq M$ . The relation on  $M'$  which is obtained by restricting  $R$  to  $M'$  is also denoted by  $R$ .

In the first half of this section, we prove that  $\text{ACFO}^=$  and  $\text{ACFO}^-$  have  $\forall\exists$ -axiomatizations in  $L_c$ . We deduce this essentially from a quantifier elimination result for  $(\mathbb{U}_{(p)}; <, \mathcal{P})$ . This is done by linking a class of structures containing  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  to another class of structures with better known model theory.

Let  $\mathbb{Z}_{(p)}$  be the usual localization of  $\mathbb{Z}$  at the prime ideal  $(p)$ . This definition still applies when  $p$  is zero, in which case  $\mathbb{Z}_{(0)} = \mathbb{Q}$ . For  $n > 0$ , let  $\mathcal{D}_n \subseteq \mathbb{Z}_{(p)}$  be the unary relation for divisibility by  $n$ . Let  $\mathcal{D}$  be  $(\mathcal{D}_n)_{n>0}$  and  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1)$  be the expansion of the ordered abelian group  $(\mathbb{Z}_{(p)}; <)$  by the family  $\mathcal{D}$  and the constants 1 and  $-1$ . Then  $(\mathbb{Z}_{(p)}; <, \mathcal{D}_n, \pm 1)$  is a structure in the language  $L_a$  extending the language of order groups by a predicate symbol for each  $n$  and constant symbols for 1 and  $-1$ . Let  $T_a$  be the class of  $L_a$ -structures  $(G; <, \mathcal{D}, \pm 1)$  such that:

- (1)  $(G; <)$  is an ordered additive abelian group; 1 is a distinguished positive element and  $-1$  is a distinguished negative element such that  $(-1) + 1 = 0$ ;
- (2) The family of unary predicate  $\mathcal{D}$  on  $G$  is defined as above replacing  $\mathbb{Z}_{(p)}$  with  $G$ ;
- (3) there is at most one prime  $l$  such that  $\neg \mathcal{D}_l(1)$ ;
- (4) if  $l$  is prime with  $\mathcal{D}_l(1)$  and  $q = l^k$  with  $k > 1$ , then for all  $\alpha \in G$ ,  $\mathcal{D}_q(\alpha)$ ;
- (5) if  $l$  is a prime such that  $\neg \mathcal{D}_l(1)$  and  $q = l^k$  with  $k \in \mathbb{N}_{\geq 1}$ , then for all  $\alpha \in G$ , there is exactly one  $r \in \{0, 1, \dots, q-1\}$  such that  $\mathcal{D}_q(\alpha + r \cdot (-1))$ ;
- (6) for all  $n > 0$  and  $\beta, \beta' \in G$  with  $\beta < \beta'$ , there is  $\alpha \in G$  with  $\beta < \alpha < \beta'$  and  $\mathcal{D}_n(\alpha)$ .

When  $p$  is prime, we define  $T_{a,p}$  by adding to the above list the property that  $\neg \mathcal{D}_p(1)$ ; when  $p$  is zero, we define  $T_{a,p}$  by adding to the above list the property that  $\mathcal{D}_l(1)$  for all prime  $l$ . Clearly,  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1)$  is a model of  $T_{a,p}$  and is uniquely  $L_a$ -embeddable into an arbitrary model of  $T_{a,p}$ . We can easily see that the classes  $T_a$  and  $T_{a,p}$  for arbitrary  $p$  have  $\forall\exists$ -axiomatizations in  $L_a$  and that  $T_a = \bigcup_p T_{a,p}$ .

**Lemma 3.1.** *The theory  $T_a$  admits quantifier elimination. For all  $p$  either prime or zero,  $T_{a,p}$  is complete.*

*Proof.* By (1) and (5) of the definition, every model of  $T_a$  is a dense regular ordered abelian group as defined in [RZ60]. By a result in [Wei81],  $T_a$  admits quantifier elimination in  $L_a$ ; a more model theoretic proof can also be easily obtained (see [vdDGn06]). For all  $p$ , an arbitrary model of  $T_{a,p}$  extends a copy of  $(\mathbb{Z}_{(p)}; <, \mathcal{D}_n, \pm 1)$  as  $L_a$ -structure. Hence,  $T_{a,p}$  is complete.  $\square$

The structure  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1)$  can be constructed from  $(\mathbb{U}_{(p)}; <, \mathcal{P})$ . The group homomorphism  $\alpha \mapsto e^{2\pi i \alpha}$  maps  $\mathbb{Z}_{(p)}$  onto  $\mathbb{U}_{(p)}$  with kernel  $\mathbb{Z}$ . We can therefore identify the underlying set of  $\mathbb{Z}_{(p)}$  with that of  $\mathbb{Z} \times \mathbb{U}_{(p)}$ . Moreover, we can equip  $\mathbb{Z} \times \mathbb{U}_{(p)}$  with an  $L_a$ -structure. Let  $a, a'$  be in  $\mathbb{U}_{(p)}$ . Define  $+$  on  $\mathbb{Z} \times \mathbb{U}_{(p)}$  by

$$(k, a) + (k', a') = \begin{cases} (k + k', aa') & \text{if } a \leq aa' \text{ in } (\mathbb{U}_{(p)}; <, \mathcal{P}), \\ (k + k' + 1, aa') & \text{otherwise.} \end{cases}$$

Let  $<$  be the lexicographic ordering on  $\mathbb{Z} \times \mathbb{U}_{(p)}$ . Let  $\mathcal{D} = (\mathcal{D}_n)_{n>0}$  be given by

$$(k, a) \in \mathcal{D}_n \text{ if and only if } a \in \mathcal{P}_n^r \text{ and } k \equiv r \pmod{n}.$$

Finally, the constants  $-1, 0$  and  $1$  on  $\mathbb{Z} \times \mathbb{U}_{(p)}$  are defined to be the pairs  $(-1, 1)$ ,  $(0, 1) \in \mathbb{Z} \times \mathbb{U}_{(p)}$  and  $(1, 1) \in \mathbb{Z} \times \mathbb{U}_{(p)}$  respectively. By construction,  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1)$  is  $L_a$ -isomorphic to  $(\mathbb{Z} \times \mathbb{U}_{(p)}; <, \mathcal{D}, \pm 1)$ .

Replacing  $\mathbb{U}_{(p)}$  with  $M$  and  $\mathbb{Z}_{(p)}$  with  $G$ , we get the definition of the  $L_a$ -**cover**  $(G; <, \mathcal{D}, \pm 1)$  of  $(M; <, \mathcal{P}) \models T_m(\forall)$  where  $T_m(\forall) = \bigcup_p T_{m,p}(\forall)$  as a class. This defines a functor  $\mathcal{F}_a$  from  $(T_m(\forall), \hookrightarrow)$  to  $([L_a], \hookrightarrow)$ .

**Lemma 3.2.** *For all  $p$ ,  $\mathcal{F}_a(T_{m,p}) \subseteq T_{a,p}$  and  $\mathcal{F}_a(T_{m,p}(\forall)) \subseteq T_{a,p}(\forall)$ . Moreover,  $\mathcal{F}_a(T_m) \subseteq T_a$  and  $\mathcal{F}_a(T_m(\forall)) \subseteq T_a(\forall)$ .*

*Proof.* To prove  $\mathcal{F}_a(T_{m,p}) \subseteq T_{a,p}$ , suppose  $(M; <, \mathcal{P}) \models T_{m,p}$  and  $(G; <, \mathcal{D}, \pm 1)$  is its  $L_a$ -cover. For each  $m > 0$ , we let

$$G_m = \{k : -m \leq k \leq m\} \times M$$

and get  $(G_m; R_+, <, \mathcal{D}, \pm 1)$  by viewing  $+$  on  $G$  as a ternary relation  $R_+$  on  $G$  and restricting  $(G; R_+, <, \mathcal{D}, \pm 1)$  to  $G_m$  in the obvious way. We note that  $(G; <, \mathcal{D}, \pm 1) \models T_{a,p}$  if and only if  $(G_m; R_+, <, \mathcal{D}, \pm 1)$  satisfy the truncated version of (1) to (5) in the definition of  $T_{a,p}$  for all  $m$ .

For all  $m > 0$ ,  $(G_m; R_+, <, \mathcal{D}, \pm 1)$  is interpretable in  $(M; <, \mathcal{P})$ . Moreover, this can be done without using parameters. Hence,  $(G; <, \mathcal{D}, \pm 1) \models T_{a,p}$  if and only if  $(M; <, \mathcal{P})$  satisfies a particular set of  $L_m$ -statements. Since  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1) \models T_{a,p}$ , this particular set of  $L_m$ -statements holds in the  $T_{m,p}$  model  $(\mathbb{U}_{(p)}; <, \mathcal{P})$ . The conclusion follows from the fact that  $T_{m,p}$  is complete.

As  $\mathcal{F}_a$  is a functor, it follows that  $\mathcal{F}_a(T_{m,p}(\forall)) \subseteq T_{a,p}(\forall)$ . The second statement is immediate.  $\square$

Conversely,  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  is interpretable in  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1)$ . The set  $\mathbb{U}_{(p)}$  can be identified with  $\mathbb{Z}_{(p)} \cap [0, 1) = \{\alpha \in \mathbb{Z}_{(p)} : 0 \leq \alpha < 1\}$  via  $a \mapsto (2\pi i)^{-1} \text{Log}(a)$ . We equip an  $L_m$ -structure on  $\mathbb{Z}_{(p)} \cap [0, 1)$ . Define  $\cdot$  on  $\mathbb{Z}_{(p)} \cap [0, 1)$  by setting

$$\alpha \cdot \beta = \begin{cases} \alpha + \beta & \text{if } \alpha + \beta < 1 \text{ in } (\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1), \\ \alpha + \beta + (-1) & \text{otherwise.} \end{cases}$$

Define  $<$  on  $\mathbb{Z}_{(p)} \cap [0, 1)$  by restricting  $<$  on  $\mathbb{Z}_{(p)}$  and  $\mathcal{P} = (\mathcal{P}_n^r)_{n>0, r \in \{0, \dots, n-1\}}$  on  $\mathbb{Z}_{(p)} \cap [0, 1)$  by setting  $\alpha \in \mathcal{P}_n^r$  if and only if  $\alpha + r \cdot 1 \in \mathcal{D}_n$ . Then the identification between  $\mathbb{U}_{(p)}$  and  $\mathbb{Z}_{(p)} \cap [0, 1)$  gives us an isomorphism of  $L_m$ -structures.

Replacing  $\mathbb{U}_{(p)}$  with  $M$ ,  $\mathbb{Z}_{(p)}$  with  $G$  and  $\mathbb{Z}_{(p)} \cap [0, 1)$  with  $G \cap [0, 1)$  defined in the obvious way, we get the definition of the  $L_m$ -**truncation**  $(M; <, \mathcal{P})$  of a  $T_a(\forall)$ -model  $(G; <, \mathcal{D}, \pm 1)$ . This defines a functor  $\mathcal{F}_m$  from  $(T_a(\forall), \hookrightarrow)$  to  $([L_m], \hookrightarrow)$ .

**Lemma 3.3.** *For all  $p$ ,  $\mathcal{F}_m(T_{a,p}) \subseteq T_{m,p}$  and  $\mathcal{F}_m(T_{a,p}(\forall)) \subseteq T_{m,p}(\forall)$ . Moreover,  $\mathcal{F}_m(T_a) \subseteq T_m$  and  $\mathcal{F}_m(T_a(\forall)) \subseteq T_m(\forall)$ .*

*Proof.* For all  $p$ , the  $L_m$ -truncation of the  $T_{a,p}$ -model  $(\mathbb{Z}_{(p)}; <, \mathcal{D}_n, \pm 1) \models T_{a,p}$  is isomorphic to  $(\mathbb{U}_{(p)}; <, \mathcal{P}) \models T_{m,p}$  and hence a model of  $T_{m,p}$ . Moreover,  $T_{a,p}$  is complete and  $L_m$ -truncations are interpretable in the corresponding  $T_{a,p}$ -models independent of the model choice. Hence,  $\mathcal{F}_m(T_{a,p}) \subseteq T_{m,p}$ . As  $\mathcal{F}_m$  is a functor,  $\mathcal{F}_m(T_{a,p}(\forall)) \subseteq T_{m,p}(\forall)$ . The second statement is immediate.  $\square$

**Lemma 3.4.** *A model of  $T_{m,p}(\forall)$  is naturally isomorphic to the  $L_m$ -truncation of its  $L_a$ -cover. Moreover the functors  $\mathcal{F}_a$  and  $\mathcal{F}_m$  are adjoint.*

*Proof.* By Lemma 3.2,  $\mathcal{F}_a(T_m(\forall)) \subseteq T_a(\forall)$ , and so the construction of  $L_m$ -truncation of  $L_a$ -cover is allowed. The first statement can then be easily checked. The second statement is not used and left to the interested reader.  $\square$

**Proposition 3.5.** *The classes  $T_m(\forall)$  and  $T_m$  are first-order axiomatizable. The theory  $T_m$  has quantifier elimination and hence has an  $\forall\exists$ -axiomatization.*

*Proof.* We show that  $T_m$  is first order axiomatizable and the statement for  $T_m(\forall)$  easily follows. Since  $T_m = \bigcup_p T_{m,p}$  and  $T_{m,p}$  is first order axiomatizable for all  $p$ , we have that  $T_m$  is closed under elementary equivalence. Suppose  $I$  is an infinite index set and for every  $i \in I$ ,  $(M_i; <, \mathcal{P})$  is the  $L_m$ -truncation of  $(G_i; <, \mathcal{D}, \pm 1) \models T_a$ . As  $(M_i; <, \mathcal{P})$  is interpretable in  $(G_i; <, \mathcal{D}, \pm 1)$  independent of the choice of  $i$ , for and ultra filter  $\mathcal{U}$  on  $I$ , we have that

$$\prod_{i \in I} (M_i; <, \mathcal{P}) / \mathcal{U} \cong_{L_m} \mathcal{F}_m \left( \prod_{i \in I} (G_i; <, \mathcal{D}, 1) / \mathcal{U} \right).$$

By the preceding two lemmas,  $T_m$  is closed under arbitrary ultra product. The desired conclusion follows by standard model theory (see Theorem 4.1.12 of [CK90]).

For the second statement, suppose  $(M; <, \mathcal{P})$  is an  $L_m$ -substructure of both  $(M_1; <, \mathcal{P}) \models T_m$  and  $(M_2; <, \mathcal{P}) \models T_m$ ,  $\varphi$  is in  $L_m(x)$  and  $\alpha$  is in  $M^m$ . By standard quantifier elimination test (see Theorem 3.1.4 of [Mar02]), we need:

$$(M_1; <, \mathcal{P}, \alpha) \models \varphi(\alpha) \Leftrightarrow (M_2; <, \mathcal{P}, \alpha) \models \varphi(\alpha).$$

By the preceding two lemmas and the functoriality of  $\mathcal{F}_a$  and  $\mathcal{F}_m$ , we can arrange that:  $(M; <, \mathcal{P})$  is the  $L_m$ -truncation of  $(G; <, \mathcal{D}, \pm 1) \models T_a(\forall)$ ,  $(M_1; <, \mathcal{P})$  is the  $L_m$ -truncation of  $(G_i; <, \mathcal{D}, \pm 1) \models T_{a,p}$  for  $i \in \{1, 2\}$  and  $(G; <, \mathcal{D}, \pm 1)$  is a common  $L_a$ -substructure of  $(G_1; <, \mathcal{D}, \pm 1)$  and  $(G_2; <, \mathcal{D}, \pm 1)$ . Since the interpretation of a  $L_m$ -truncation of a model of  $T_a(\forall)$  inside that model is independent of the choice of the model, there is  $\psi(x) \in L_m(x)$  such that for all  $\beta \in M_i$  and  $i \in \{1, 2\}$ ,

$$(M_i; <, \mathcal{P}, \beta) \models \varphi(\beta) \Leftrightarrow (G_i; <, \mathcal{D}, \pm 1, \beta) \models \psi(\beta)$$

Therefore our problem reduces to showing that:  $(G_1; <, \mathcal{D}, \pm 1, \alpha) \models \psi(\alpha) \Leftrightarrow (G_2; <, \mathcal{D}, \pm 1, \alpha) \models \psi(\alpha)$ . This follows from quantifier elimination of  $T_a$ .  $\square$

*Proof of Theorem 1.2, part 1.* We show that  $\text{ACFO}^+$  and  $\text{ACFO}^-$  have  $\forall\exists$ -axiomatization in  $L_c$ . We note that  $(F; <, \mathcal{P}) \models \text{ACFO}^-$  if and only if  $F \models \text{ACF}$ ,  $(F^\times; <, \mathcal{P}) \models T_m$  and  $\text{char}(F) = p \Leftrightarrow \neg \mathcal{P}_p^0(1)$ . For  $\text{ACFO}^+$ , we replace  $T_m$  with  $T_m(\forall)$ . The conclusion hence follows from the preceding proposition.  $\square$



In the second half of this section, we show that ACFO has a  $\forall\exists$ -axiomatization in  $L_c$ . This needs a further understanding of the notion of *multiplicative largeness*. In the rest of the section,  $F$  is an algebraically closed field,  $V \subseteq F^m$  is a variety and  $V^\times = V \cap (F^\times)^m$ . The multiplicative group  $(F^\times)^m$  has underlying set  $(F^\times)^m$  and multiplication given by  $ab = (a_1b_1, \dots, a_mb_m)$  for  $a, b \in (F^\times)^m$ .

**Lemma 3.6.** *If  $M$  is an algebraic subgroup of the multiplicative group  $(F^\times)^m$  then  $M$  is the set of elements of  $(F^\times)^m$  satisfying a system of polynomial equations each of which has the form  $x_1^{k_1} \dots x_m^{k_m} = 1$  with  $k_1, \dots, k_m \in \mathbb{Z}$ .*

*Proof.* This is Corollary 3.2.15 in [BG06]. There is an extra assumption that the field is of characteristic 0 in the given reference but the proof of this particular result goes through even without this assumption.  $\square$

**Corollary 3.7.** *The variety  $V$  is multiplicatively large if and only if for some (equivalently for all)  $b \in V^\times$ , the only definable subgroup of  $(F^\times)^m$  containing  $b^{-1}V^\times$  is  $(F^\times)^m$ .*

*Proof.* For the forward direction, let  $V$  be multiplicatively large and  $M$  is a definable subgroup of  $(F^\times)^m$  containing  $b^{-1}V^\times$  for an arbitrary  $b \in V^\times$ . By a well known result (see Lemma 7.4.9 of [Mar02]),  $M$  is an algebraic group. Hence  $M$  is the set of elements of  $(F^\times)^m$  satisfying a system of polynomial equations as in the preceding lemma. Suppose  $x_1^{k_1} \dots x_m^{k_m} = 1$  with  $k_1, \dots, k_m \in \mathbb{Z}$  is one of the equation in the system. Then all  $a \in V^\times$  satisfies:

$$x_1^{k_1} \dots x_m^{k_m} = b_1^{k_1} \dots b_m^{k_m}.$$

As  $V$  is multiplicatively large,  $k_1 = \dots = k_m = 0$ . Thus,  $M = (F^\times)^m$ .

The reverse direction is straight forward noting that  $V$  not multiplicatively large implies that for some  $k_1, \dots, k_m \in \mathbb{Z}$  not all zero,  $b^{-1}V^\times$  for any  $b \in V^\times$  satisfies  $x_1^{k_1} \dots x_m^{k_m} = 1$  which defines a nontrivial subgroup of  $(F^\times)^m$ .  $\square$

Suppose  $M$  is a multiplicative group and  $X_1, \dots, X_n$  are subset of  $M$ . We set  $X_1 \dots X_n = \{a_1 \dots a_n : a_i \in X_i \text{ for } 1 \leq i \leq n\}$ . Moreover, if  $X_1 = \dots = X_n = X$ , then we set  $X^{\cdot k} = X_1 \dots X_n$ .

**Lemma 3.8** (Zilber's Indecomposability Theorem). *Let  $(M; \cdot, \dots)$  be a multiplicative group of finite Morley rank and  $(X_i)_{i \in I}$  be a collection indecomposable definable subsets of  $M$  containing 1. Then there is  $k > 0$  and  $i_1, \dots, i_k \in I$  with possible repetition such that  $X_{i_1} \dots X_{i_k}$  is the group generated by  $(X_i)_{i \in I}$ .*

*Proof.* See Theorem 7.3.2 of [Mar02].  $\square$

**Corollary 3.9.** *There is  $k > 0$  such that  $(b^{-1}V^\times)^{\cdot k} = (b^{-1}V^\times)^{\cdot k+1}$  for some  $b \in V^\times$ . Moreover,  $V$  is multiplicatively large if and only if for such  $k$  we also have that  $(b^{-1}V^\times)^{\cdot k} = (F^\times)^m$ .*

*Proof.* The first is immediate from the preceding lemma noting that  $1 \in b^{-1}V$  and  $b^{-1}V$  is indecomposable (see Exercise 7.6.13 of [Mar02]). The second statement follows from Corollary 3.7 since  $(b^{-1}V^\times)^{\cdot k}$  as in the first statement is the smallest definable subgroup of  $(F^\times)^m$  containing  $b^{-1}V$ .  $\square$

Let  $M$  be a structure in a language  $L$ . Recall that a family  $(X_s)_{s \in S}$  of subset of  $M^m$  is *definable* if  $S \subseteq M^n$  for some  $n$  is definable and there is definable  $X \subseteq M^{m+n}$  such that for all  $s \in S$ ,  $X_s = \{a \in M^m : (a, s) \in X\}$ . If  $M \preceq M'$ , then we define  $(X_s)_{s \in S}(M')$  to be the family  $(X'_{s'})_{s' \in S'}$  where  $S' = S(M')$  and for  $s \in S$ ,  $X'_s = \{a' \in (M')^m : (s', s') \in X'\}$  with  $X' = X(M')$ .

**Lemma 3.10.** *Let  $(X_s)_{s \in S}$  be an  $L_r$ -definable family of subsets of  $F^m$ . Then the set  $\{s \in S : X_s \text{ is a variety}\}$  is definable in  $L_r$ .*

*Proof.* See Theorem 10.2.1 of [Joh16].  $\square$

**Lemma 3.11.** *Let  $(V_s)_{s \in S}$  be an  $L_r$ -definable family of subsets of  $F^m$  which are varieties over  $F$ . Then  $\{s \in S : V_s \text{ is multiplicatively large}\}$  is definable in  $L_r$ .*

*Proof.* Let  $(V_s)_{s \in S}$  be as given. We first prove that if  $F \preceq F'$  then the family  $(V'_{s'})_{s' \in S'} = (V_s)_{s \in S}(F')$  is a family of varieties over  $F'$ . We note that if  $F' \preceq F''$ , then  $V'_{s'}$  is a variety over  $F'$  if and only if  $V'_{s'}(F'')$  is a variety over  $F''$ . Hence, by extending  $F'$  further if needed, we can arrange that  $F'$  is sufficiently saturated. From the preceding lemma, the set

$$S'_v = \{s' \in S' : V'_{s'} \text{ is a variety}\}$$

is definable. Moreover, any automorphism of  $F'$  fixing  $F$  also fixes  $S'_v$ , so  $S'_v$  is definable over  $F$ . Suppose  $S' \setminus S'_v \neq \emptyset$ . As  $F \preceq F'$ , there is  $s \in (S' \setminus S'_v) \cap F^n$ . Since  $V_s$  is a variety over  $F$ ,  $V'_s = V_s(F')$  is a variety over  $F'$ , contradiction.

For  $n > 0$ , let  $S_k$  be the set of  $s \in S$  such that for some  $b \in V_s^\times$  we have  $(b^{-1}V_s^\times)^k = (b^{-1}V_s^\times)^{k+1}$ . Clearly,  $S_k$  is definable for all  $n > 0$  and  $S = \bigcup_{k>0} S_k$  by the first statement of Corollary 3.9. Suppose  $F \preceq F'$  and  $(V'_{s'})_{s' \in S'} = (V_s)_{s \in S}(F')$ . As  $(V'_{s'})_{s' \in S'}$  is a family of varieties over  $F'$ , a similar argument yields  $S' = \bigcup_{k>0} S'_k$  with  $S'_k$  defined similarly. It is easy to see that  $S'_k = S_k(F')$ . Therefore,

$$S(F') = \bigcup_{k>0} S_k(F').$$

A standard compactness argument gives us  $S = S_k$  for some  $k > 0$ . The desired conclusion then follows from the second statement of Corollary 3.9.  $\square$

**Corollary 3.12.**  *$(X_s)_{s \in S}$  be an  $L_r$ -definable family of subsets of  $F^m$ . Then the set  $\{s \in S : X_s \text{ is a multiplicatively large variety over } F\}$  is definable in  $L_r$ .*  $\square$

*Proof of Theorem 1.2, part 2.* We show that ACFO has a  $\forall\exists$  axiomatization. Suppose  $(F; <, \mathcal{P}) \models \text{ACFO}^-$ . We will write  $b < b'$  for  $b, b' \in (F^\times)^m$  if  $b_i < b'_i$  as  $i$  ranges over  $\{1, \dots, m\}$ . From the preceding corollary and quantifier elimination of ACF, for all  $n$  and all  $\varphi \in L_r(x, y)$ , there is a quantifier free formula  $\psi_\varphi \in L_r(y)$  which defines

$$\{s \in F^n : \varphi(x, s) \text{ defines a multiplicatively large variety}\}.$$

On the other hand, quantifier elimination for ACF implies that for every variety  $V$  there is  $n$ , a quantifier free formula  $\varphi \in L_r(x, y)$ ,  $s \in F^n$  such that  $V$  is the set defined by  $\varphi(x, s)$ . As a consequence,  $(F; <, \mathcal{P}) \models \text{ACFO}$  if and only if for all choices of  $m, n$  and a quantifier free formula  $\varphi \in L_r(x, y)$  we have that for all  $s \in F^n$  with  $\psi_\varphi(s)$ , for all  $b, b' \in (F^\times)^m$ , with  $b < b'$ , there is  $a \in (F^\times)^m$  with  $\varphi(a, s)$  and  $b < a < b'$ . The desired conclusion follows.  $\square$

## 4. LOGICAL TAMENESS

In this section, we use the following conventions in addition to those introduced in the first paragraphs of the preceding two sections. Let  $F$  range over algebraically closed fields,  $V$  range over quasi-affine subvarieties of  $F^m$  and  $V^\times = V \cap (F^\times)^m$ . The model-theoretic algebraic (definable) closure operators in  $L_r$ ,  $L_m$  and  $L_c$  are denoted by  $\text{acl}_r$ ,  $\text{acl}_m$  and  $\text{acl}_c$  ( $\text{dcl}_r$ ,  $\text{dcl}_m$  and  $\text{dcl}_c$ ). We will use the term *algebraic independence* in the field theoretic sense. If  $M$  is a multiplicative group,  $a$  is a (possibly infinite) tuple of elements in  $M$ ,  $A$  is a subset of  $M$ , let  $\langle a \rangle$  and  $\langle A \rangle$  be the subgroup of  $M$  generated by the terms of  $a$  and the elements of  $A$  respectively.

A model  $(F; <, \mathcal{P})$  of ACFO is in the strict sense not the amalgamation of  $F$  and  $(F^\times; <, \mathcal{P})$  over  $F^\times$  as  $0 \notin F^\times$ . It is convenient to replace  $F$  by a structure expanding the multiplicative group  $F^\times$  with relations “remembering” the additive structure. For each choice of  $m, n$  we let  $\mathcal{A}_{m,n} \subseteq (F^\times)^{m+n}$  be the set of  $(a, b) \in (F^\times)^m \times (F^\times)^n$  such that

$$a_1 + \cdots + a_m = b_1 + \cdots + b_n.$$

We do allow  $m, n$  to be zero, in which case the corresponding side is zero. Let  $(F^\times; \mathcal{A})$  be the expansion of  $F^\times$  by adding the collection of relations  $\mathcal{A} = (\mathcal{A}_{m,n})$ . Then  $(F^\times; \mathcal{A})$  is a structure in a language  $L_r^\times$  extending  $L_g$  by adding a predicate symbol for each choice of  $m, n \in \mathbb{N}$ . We call  $(F^\times; \mathcal{A})$  the  $L_r^\times$ -**reduct** of  $F$ . The construction defines a functor  $\mathcal{F}_r^\times$  from the category  $(\text{ACF}, \hookrightarrow)$  to the category  $([L_c^\times], \hookrightarrow)$ . We set  $\text{ACF}^\times = \mathcal{F}_r^\times(\text{ACF})$  and  $\text{ACF}_p^\times = \mathcal{F}_r^\times(\text{ACF}_p)$ . The following observation is an easy consequence of the quantifier elimination of ACF:

**Lemma 4.1.** *Suppose  $\varphi$  is in  $L_r(x)$ , there is a formula  $\varphi^\times$  in  $L_r^\times(x)$  such that if  $(F^\times; \mathcal{A})$  is the  $L_c^\times$ -reduct of  $F$  and  $a \in (F^\times)^m$ , then  $\varphi(a)$  holds in  $F$  if and only if  $\varphi^\times(a)$  holds in  $(F^\times; \mathcal{A})$ .*

The classes  $\text{ACF}^\times$  and  $\text{ACF}_p^\times$  for arbitrary  $p$  enjoy all the first order properties of ACF and  $\text{ACF}_p$  for arbitrary  $p$ .

**Lemma 4.2.** *The classes  $\text{ACF}^\times$ ,  $\text{ACF}_p^\times$  admit  $\forall\exists$ -axiomatizations. Moreover,  $\text{ACF}^\times$  has quantifier elimination and for  $p$  prime or zero,  $\text{ACF}_p^\times$  is complete.*

*Proof.* For the first statement, we will only need to show that ACF is first-order axiomatizable as the existence of a  $\forall\exists$ -axiomatizations follows from quantifier elimination and the proof for  $\text{ACF}_p$  is similar. It suffices to check that  $\text{ACF}^\times$  is closed under elementary equivalence and ultraproducts.

Let  $(F^\times; \mathcal{A})$  be a model of  $\text{ACF}^\times$ . We note that  $F$  is parameter-free interpretable in  $(F^\times; \mathcal{A})$ . Hence,  $(F^\times; \mathcal{A})$  is isomorphic to the  $L_r^\times$ -reduct of an algebraically closed field which is parameter-free interpretable in  $(F^\times; \mathcal{A})$ . Moreover, the isomorphism in the preceding statement is also parameter-free interpretable in  $(F^\times; \mathcal{A})$ . All these are first-order expressible in  $L_r^\times$ . Thus,  $\text{ACF}^\times$  is closed under elementary equivalence.

As the  $L_r^\times$ -reduct  $(F^\times; \mathcal{A})$  of  $F \models \text{ACF}$  is interpretable in  $F$ , an ultra-product  $\prod_{i \in I} (F_i^\times; \mathcal{A})/\mathcal{U}$  of models of  $\text{ACF}^\times$  is isomorphic to the  $L_r^\times$ -reduct of the ultra-product  $\prod_{i \in I} F_i/\mathcal{U}$ . Hence,  $\text{ACF}^\times$  is closed under ultraproducts.

The quantifier elimination of  $\text{ACF}^\times$  and the completeness of  $\text{ACF}_p^\times$  follow easily from those of ACF and  $\text{ACF}_p$ .  $\square$

Let  $(F; <, \mathcal{P})$  be a model of  $\text{ACFO}^-$  and  $(F^\times; \mathcal{A})$  be the associated  $L_r^\times$ -structure of  $F$ . Then  $(F^\times; \mathcal{A}, <, \mathcal{P})$  is a structure in a language  $L_c^\times$  which is the union of  $L_r^\times$  and  $L_m$ . We call  $(F^\times; \mathcal{A}, <, \mathcal{P})$   $L_c^\times$ -reduct of  $(F; <, \mathcal{P})$ . This defines a functor  $\mathcal{F}_c^\times$  from  $(\text{ACFO}^-, \hookrightarrow)$  to  $([L_c^\times], \hookrightarrow)$ . We set  $\text{ACFO}^\times = \mathcal{F}_c^\times(\text{ACFO})$  and  $\text{ACFO}_p^\times = \mathcal{F}_c^\times(\text{ACFO}_p)$ . With a similar proof as the preceding lemma, we have:

**Lemma 4.3.** *The classes  $\text{ACFO}^\times$ ,  $\text{ACFO}_p^\times$  are first-order axiomatizable.*

We will deduce the model completeness of  $\text{ACFO}$  from that of  $\text{ACFO}^\times$ . The notions introduced below are essentially the translation of the notions of *generic* in [CP98], [Tsu01] and [SS12]. Let  $(G; \dots)$  be a structure expanding an additive abelian group. We say a definable  $X \subseteq G^m$  **permits linear independence** if there is an elementary extension  $(G'; \dots)$  of  $(G; \dots)$  and  $a' \in X' = X(M')$  which is linearly independent over  $G$ . When  $(M; \dots)$  is structure expanding a multiplicative group, we define the notion of **permitting multiplicative independence** likewise replacing linear independence with multiplicative independence. The following is an easy observation:

**Lemma 4.4.** *Suppose  $(F^\times; \mathcal{A}) \models \text{ACF}^\times$ . Then  $V^\times \subseteq (F^\times)^m$  permits multiplicative independence if and only if  $V$  is multiplicatively large.*

*Proof.* The forward implication is straight forward from the definition and the backward implication follows easily from compactness.  $\square$

**Corollary 4.5.** *Suppose  $(F^\times; \mathcal{A}) \models \text{ACF}^\times$  and  $X \subseteq (F^\times)^m$  is definable  $L_r^\times$ . Then  $X$  permits multiplicative independence if and only if there is multiplicatively large  $V \subseteq F^m$  such that  $V^\times \subseteq X$ .*

*Proof.* Suppose  $(F^\times; \mathcal{A})$  is the  $L_r^\times$ -reduct of  $F$  and  $X$  is as given. Then  $X$  is a restriction of an  $L_r$ -definable set in  $F$ . By quantifier elimination of  $\text{ACF}$ ,

$$X = V_1^\times \cup \dots \cup V_k^\times \text{ where } V_i^\times = V_i \cap (F^\times)^m$$

and  $V_i$  is a quasi-affine variety for  $i \in \{1, \dots, k\}$ . If  $X$  permits multiplicative independence, then  $V_i^\times$  permits multiplicative independence for some  $i \in \{1, \dots, k\}$ . The conclusion follows from the preceding lemma. The backward direction is clear from the preceding lemma.  $\square$

We will obtain a similar characterization of definable subsets of a model of  $T_{m,p}$  permitting multiplicative independence. Let  $(G; <, \mathcal{D}, \pm 1) \models T_{a,p}$ . We can view  $G^m$  as an additive group in an obvious way. For  $\alpha, \beta \in G^m$ , we write  $\alpha < \beta$  if  $\alpha_i < \beta_i$  for  $i$  in  $\{1, \dots, m\}$ . Suppose  $q = p^k$  for  $k \geq 0$ . We call  $H \subseteq G^m$  a  **$q$ -hyper-rectangle** if there are  $\beta < \beta' \in G^m$  and  $\varepsilon \in G^m$  such that

$$H = \{\alpha \in G^m : \beta < \alpha < \beta' \text{ and } \alpha + \varepsilon \in (\mathcal{D}_q)^m\}.$$

In parallel fashion, let  $(M; <, \mathcal{P})$  be a model of  $T_{m,p}$ . Multiplication on  $M^m$  is defined in an obvious way. We define  $a < b$  for  $a, b \in M^m$  in a similar fashion as above. A set  $H \subseteq M^m$  is a  **$q$ -hyper-arc** if there are  $b < b' \in M^m$  and  $e \in M^m$  such that  $b < be, b' < b'e$  and

$$H = \{a \in M^m : b < a < b', \text{ and } ae \in (\mathcal{P}_q^0)^m\}.$$

In the special case where  $k = 0$  and  $q = 1$ , we simply call  $H$  a **hyper-arc**.

**Lemma 4.6.** *Suppose  $(G; <, \mathcal{D}, \pm 1) \models T_{a,p}$  and  $Y \subseteq G^m$  is definable in  $L_a$ . Then  $X$  permits linear independence if and only if there is a  $q$ -hyper-rectangle  $H \subseteq Y$ .*

*Proof.* This follows from section 3 of [Tow13].  $\square$

**Corollary 4.7.** *Suppose  $(M; <, \mathcal{P}) \models T_{m,p}$  and  $X \subseteq M^m$  is definable in  $L_m$ . Then  $X$  permits multiplicative independence if and only if there is a  $q$ -hyper-arc  $H \subseteq X$ .*

*Proof.* Throughout the proof, suppose  $(M; <, \mathcal{P})$  and  $X$  are as given. Then by Lemma 3.2,  $(M; <, \mathcal{P})$  has  $L_a$ -cover  $(G; <, \mathcal{D}, \pm 1) \models T_{a,p}$ . From the construction of  $L_m$ -truncation and Lemma 3.4, there is a bijection

$$\iota : M \rightarrow \{\alpha \in G : 0 \leq \alpha < 1\}$$

which induces an  $L_m$ -isomorphism between  $(M; <, \mathcal{P})$  and the  $L_m$ -truncation of  $(G; <, \mathcal{D}, \pm 1)$ . We also denote by  $\iota$  the induced map on  $M^m$  for all  $m$ . In view of the preceding lemma, it suffices to show the following:

- (1)  $X \subseteq M^m$  permits multiplicative independence in  $(M; <, \mathcal{P})$  if and only if  $\iota(X) \subseteq G^m$  permits linear independence in  $(G; <, \mathcal{D}, \pm 1)$
- (2)  $H \subseteq M^m$  is a hyper-arc in  $(M; <, \mathcal{P})$  if and only if  $\iota(H)$  is a hyper-rectangle in  $(G; <, \mathcal{D}, \pm 1)$ .

We prove the forward direction of (1) and omit the backward direction as they are very similar. Suppose  $X$  permits multiplicative independence. Then there is elementary extension  $(M'; <, \mathcal{P})$  of  $(M; <, \mathcal{P})$  and  $\alpha \in X(M')$  multiplicative independent over  $M$ . Again from the construction of  $L_m$ -truncation and Lemma 3.4,  $(M'; <, \mathcal{P})$  has  $L_a$ -cover  $(G'; <, \mathcal{D}, \pm 1) \models T_{a,p}$  and there is a bijection

$$\iota' : M' \rightarrow \{\alpha' \in G' : 0 \leq \alpha' < 1\}$$

which induces an  $L_m$ -isomorphism between  $(M'; <, \mathcal{P})$  and the  $L_m$ -truncation of  $(G'; <, \mathcal{D}, \pm 1)$ . As  $\mathcal{F}_a$  is a functor from  $(T_{m,p}, \hookrightarrow)$  to  $(T_{a,p}, \hookrightarrow)$  and  $T_a$  admit quantifier elimination, we have that

$$(G; <, \mathcal{D}, \pm 1) \leq_{L_a} (G'; <, \mathcal{D}, \pm 1)$$

The same formula in  $L_m$  defines  $X$  in  $(M; <, \mathcal{P})$  and  $X(M')$  in  $(M'; <, \mathcal{P})$ . Hence, the same formula in  $L_m$  defines  $\iota(X)$  in the  $L_m$ -truncation of  $(G; <, \mathcal{D}, \pm 1)$  and  $\iota'(X(M'))$  in  $L_m$ -truncation of  $(G'; <, \mathcal{D}, \pm 1)$ . As the interpretation of the  $L_m$ -truncation is independent of the choice of the model, the same formula in  $L_a$  defines  $\iota(X)$  in  $(G; <, \mathcal{D}, \pm 1)$  and  $\iota'(X(M'))$  in  $(G'; <, \mathcal{D}, \pm 1)$ . Hence,

$$\iota'(X(M')) = \iota(X)(G').$$

Therefore  $\alpha' = \iota'(a')$  is in  $\iota(X)(G')$ . Suppose  $\alpha'$  has  $k_1\alpha'_1 + \dots + k_m\alpha'_m = \gamma$  with  $\gamma \in G$ . Let  $\delta \in G$  be the unique element such that  $\gamma = \delta + k \cdot 1$  and  $0 \leq \delta < 1$ . We can easily check that  $(a'_1)^{k_1} \dots (a'_m)^{k_m} = \iota^{-1}(\delta)$ . This implies that  $k_1 = \dots = k_m = 0$  and so  $\alpha'$  is linear indendent over  $G$ .

To show (2), we note that if  $H \subseteq \{\alpha \in G^m : 0 \leq \alpha_i < 1 \text{ for } 1 \leq i \leq m\}$  and  $H$  is a hyper-rectangle, we can find  $\beta, \beta' \in G^m$  and  $\varepsilon \in G^m$  as in the definition but moreover with

$$0 \leq \beta_i < \beta'_i < 1 \text{ and } 0 < \beta_i + \varepsilon_i < \beta'_i + \varepsilon_i < 1 \text{ for } i \in \{1, \dots, m\}.$$

The desired  $\varepsilon$  can be chosen as a model of  $T_{a,p}$  is regularly dense (see [Tow13] for details). The checking (2) is then straight forward from the definitions.  $\square$

**Proposition 4.8.** *The theory  $\text{ACFO}^\times$  is model complete and therefore has a  $\forall\exists$ -axiomatization.*

*Proof.* Let  $(F^\times; \mathcal{A}, <, \mathcal{P})$  and  $((F')^\times; \mathcal{A}, <, \mathcal{P})$  be arbitrary models of  $\text{ACFO}^\times$  such that the former is an  $L_c^\times$ -substructure of the latter. By a standard test [Mar02, Exercise 3.4.12], it suffices to show that the former is existentially closed in the latter. We can arrange that  $(F^\times; \mathcal{A}, <, \mathcal{P})$  and  $((F')^\times; \mathcal{A}, <, \mathcal{P})$  are respectively the  $L_c^\times$ -reduct of models  $(F; <, \mathcal{P})$  and  $(F'; <, \mathcal{P})$  of  $\text{ACFO}$  such that  $(F; <, \mathcal{P})$  is an  $L_c$ -substructure of  $(F'; <, \mathcal{P})$ .

Continuing with the setting above, we will reduce the problem to showing that for an arbitrary  $m$ , an  $L_r^\times$ -definable  $X \subseteq (F^\times)^m$ , and an  $L_m$ -definable set  $Y \subseteq (F^\times)^m$ ,

$$\text{if } X' = X(F'), Y' = Y(F') \text{ and } X' \cap Y' \neq \emptyset \text{ then } X \cap Y \neq \emptyset.$$

For our purpose, if  $\varphi(x) \in L_c^\times(x)$  quantifier free and defining a non empty set in  $((F')^\times; \mathcal{A}, <, \mathcal{P})$ , we need to show that  $\varphi(x)$  defines a non-empty set in  $(F^\times; \mathcal{A}, <, \mathcal{P})$ . As  $\varphi$  is quantifier free, it is a disjunction of conjunctions of atomic formulas. We can easily reduce to the case where  $\varphi$  is just a conjunction of atomic formulas. The only operations here is multiplication which belong to both  $L_r^\times$  and  $L_m$ , so it is easy to choose the corresponding  $X, Y$  that provide the desired reduction.

With the same assumptions as in the previous paragraph, we reduce the problem further to the case where  $X, Y$  permit multiplicative independence. Let  $a'$  be in  $X' \cap Y'$  and  $M$  be the subgroup of  $(F')^\times$  generated by  $F^\times$  and  $a'$ . Then the finitely generated group  $M/F^\times$  is a subgroup of  $(F')^\times/F^\times$  and hence torsion free. Hence,  $M/F^\times$  is isomorphic to  $\mathbb{Z}^n$  as a group for some  $n$ . As a consequence, we can find an  $n$ -tuple  $b'$  in  $M$  mutliplicative independent over  $F$  such that

$$a' = f(b') \text{ where } f = (f_1, \dots, f_m) \text{ and } f_i \text{ is of the form } cy_1^{k_1} \dots y_n^{k_n}$$

with  $c \in F^\times$  and  $k_1, \dots, k_n \in \mathbb{Z}$  for  $i \in \{1, \dots, m\}$ . Replace  $m$  with  $n$ ,  $X$  with  $f^{-1}(X)$ ,  $Y$  with  $f^{-1}(Y)$ ,  $X'$  with  $f^{-1}(X')$ ,  $Y'$  with  $f^{-1}(Y')$ , and  $a'$  with  $b'$  we achieve the desired reduction.

Finally, we address the above special case. By Corollary 4.5 and Corollary 4.7, it suffices to show for an arbitrary multiplicatively large variety  $V \subseteq (F^\times)^m$  and an arbitrary  $q$ -hyper arc  $H$  with  $q = p^k, k \geq 1$  that  $V \cap H$  is non-empty. By multiplicatively translating  $V, H$  we can reduce to the case where

$$H = \{a \in (F^\times)^m : b < a < b', \text{ and } a \in (\mathcal{P}_q^0)^m\}$$

with  $b < b'$  elements in  $(F^\times)^m$ . Shrinking  $H$  if needed we can arrange that  $b$  and  $b'$  are in  $\mathcal{P}_q^0$ . Replacing  $V$  with  $\text{Frob}^{-k}(V)$  and  $H$  with the set

$$\{a \in (F^\times)^m : a^q \in H \text{ and } a \leq a^2 \leq \dots \leq a^q\},$$

we can arrange that  $H$  is a hyper-arc. Then  $V \cap H$  is non-empty following the assumption of genericity.  $\square$

*proof of Theorem 1.3, part 1.* We show that  $\text{ACFO}$  is model complete. Suppose  $(F_1; <, \mathcal{P}), (F_2; <, \mathcal{P})$  are models of  $\text{ACFO}$  and the latter  $L_c$ -extends the former. We have that  $(F_1; <, \mathcal{P})$  is interpretable in its  $L_c^\times$ -reduct  $(F_1^\times; \mathcal{A}, <, \mathcal{P})$ . A similar statements holds substituting by  $(F_2; <, \mathcal{P})$  and  $(F_2^\times; \mathcal{A}, <, \mathcal{P})$ . Moreover, the interpretations can be chosen to preserve the inclusion between  $(F_1; <, \mathcal{P})$  and  $(F_2; <, \mathcal{P})$ . The conclusion then follows from the preceding proposition.  $\square$

Next, we will show that every model of  $\text{ACFO}^-$  can be  $L_c$ -embedded into a model of  $\text{ACFO}$ . This will be deduced from a stronger result about  $\text{ACFO}_p^\times$  for  $p$  either prime or zero. We need a quantifier elimination result for  $\mathbb{U}_{(p)}$ .

Let  $L_g$  be the language of multiplicative groups and  $T_{g,p}$  with  $p$  either prime or zero be the class of structures  $M$  in  $L_g$  such that:

- (1)  $M$  is a divisible abelian group;
- (2) for all  $n > 0$ , if  $a, b \in M$  both have order  $n$ , then there is  $k \in \{1, \dots, n\}$  such that  $a^k = b$ ;
- (3) For all  $n > 0$ , the number of  $n$ -th roots of 1 is  $n/p^k$  where  $p^k$  is the highest power of  $p$  dividing  $n$ . (When  $p$  is zero, the number of  $n$ -th roots of 1 is exactly  $n$  because the highest power of 0 dividing  $n$  is  $0^0 = 1$ ).

We can easily see that  $T_{g,p}$  is  $\forall\exists$ -axiomatizable for arbitrary  $p$  and  $\mathbb{U}_{(p)} \models T_{g,p}$ . Hence, if  $F \models \text{ACF}_p$ , then  $F^\times \models T_{g,p}$  and if  $(M; <, \mathcal{P}) \models T_{m,p}$ , then  $M \models T_{g,p}$ .

**Lemma 4.9.** *For  $p$  either prime or 0, the theory  $T_{g,p}$  has quantifier elimination and is complete.*

*Proof.* In this proof, let  $M$  and  $M'$  be models of  $T_{g,p}$  with  $M'$   $\kappa^+$ -saturated where  $\kappa = |M|$  and let  $f$  be an  $L_g$ -partial isomorphism from  $M$  to  $M'$  (in other words,  $f$  is an  $L_g$ -isomorphism from an  $L_g$ -substructure of  $M$  to an  $L_g$ -substructure of  $M'$ ). Using a standard quantifier elimination test, it suffices to show that either  $\text{Dom}(f) = M$  or there is a  $L_g$ -partial-isomorphism from  $M$  to  $M'$  which properly extends  $f$ .

Suppose  $\text{Dom}(f) \neq M$ . In this paragraph and the next three, we will address four cases which cover all possibility. Suppose  $\text{Dom}(f)$  is not a group. Extend  $f$  by mapping  $ab^{-1}$  to  $f(a)(f(b))^{-1}$ . It is clear that this satisfies the desired properties.

Suppose  $l$  is a prime and  $a \in M \setminus \text{Dom}(f)$  an  $l$ -th root of unity. Clearly,  $l \neq p$ . By the property (2) in the definition of  $T_{g,p}$ ,  $\text{Dom}(f)$  and  $\text{Im}(f)$  contain no  $l$ -root of unity other than 1. Choose  $a'$  a root of unity in  $M' \setminus \text{Im}(f)$ , which must exist because of property (3) in the definition of  $T_{g,p}$ . We can verify that

$$a^k b \mapsto (a')^k f(b) \text{ for } k \in \mathbb{Z} \text{ and } b \in \text{Dom}(f)$$

defines an  $L_g$ -partial isomorphism which extends  $f$ .

Suppose  $\text{Dom}(f)$  contains all roots of unity of prime order in  $M$ ,  $l$  is a prime and  $a \in M \setminus \text{Dom}(f)$  is such that  $a^l \in \text{Dom}(f)$ . As any other  $l$ -th root of  $a^l$  multiplicatively differs from  $a$  by an  $l$ -th root of unity,  $\text{Dom}(f)$  contains no  $l$ -th root of  $a^l$ . Hence,  $\text{Im}(f)$  contains no  $l$ -th root of  $f(a^l)$ . Choose  $a'$  an  $l$ -th root of  $f(a^l)$  which must exist as  $M$  is divisible. Again, we can verify that

$$a^k b \mapsto (a')^k f(b) \text{ for } k \in \mathbb{Z} \text{ and } b \in \text{Dom}(f)$$

defines an  $L_g$ -partial isomorphism which extends  $f$ .

Suppose  $\text{Dom}(f)$  is divisibly closed in  $M$ , and  $a \in M \setminus \text{Dom}(f)$ . Choose  $a'$  in  $M'$  multiplicatively independent over  $\text{Im}(f)$  which must exist because  $M'$  is  $|M|^+$ -saturated. Again, we can verify that

$$a^k b \mapsto (a')^k f(b) \text{ for } k \in \mathbb{Z} \text{ and } b \in \text{Dom}(f)$$

defines an  $L_g$ -partial isomorphism which extends  $f$ .

When  $n$  coprime with  $p$ , the group of  $n$ -roots of 1 in a model of  $T_{g,p}$  is cyclic of size  $n$  by (2) and (3). Hence, any model of  $T_{g,p}$  is a  $L_g$ -extension of a copy of  $\mathbb{U}_{(p)}$  which is a model of  $T_{g,p}$ . Thus,  $T_{g,p}$  is complete.  $\square$

**Proposition 4.10.** *If  $(M; \mathcal{A}, <, \mathcal{P})$  is an  $L_c^\times$ -structure with  $(M; \mathcal{A}) \models \text{ACF}_p^\times(\forall)$  and  $(M; <, \mathcal{P}) \models T_{m,p}(\forall)$ , then  $(M; \mathcal{A}, <, \mathcal{P})$  can be  $L_c^\times$ -embedded into a model of  $\text{ACFO}_p^\times$ .*

*Proof.* Throughout the proof, suppose  $(M; \mathcal{A}, <, \mathcal{P})$  is as in the statement of the lemma. We first show that it is can be  $L_c^\times$ -embedded into  $(F^\times; \mathcal{A}, <, \mathcal{P})$  such that  $(F^\times; \mathcal{A}) \models \text{ACF}_p^\times$  and  $(F^\times; <, \mathcal{P}) \models T_{m,p}(\forall)$ . Clearly, there is  $(F^\times; \mathcal{A}) \models \text{ACF}_p^\times$  extending  $(M; \mathcal{A})$  as an  $L_r^\times$ -structure. We can also find  $(N, <, \mathcal{P}) \models T_{m,p}$  which is  $|M'|^+$ -saturated and extend  $(M; <, \mathcal{P})$  as an  $L_m$ -structure. By the remark just before the preceding proposition, we have that:

$$F^\times \models T_{g,p} \text{ and } N \models T_{g,p}.$$

Since  $T_{g,p}$  has quantifier elimination and  $N$  is  $|M'|^+$ -saturated,  $M'$  can be  $L_g$ -embedded into  $N$  over  $M$ . We can then equip  $M'$  with relations  $<$  and  $\mathcal{P}$  by pullback. It is easy to see that  $(M'; \mathcal{A}, <, \mathcal{P})$  has the desired properties by construction.

We next show that  $(M; \mathcal{A}, <, \mathcal{P})$  can be  $L_c^\times$ -embedded into  $(F^\times; \mathcal{A}, <, \mathcal{P})$  with

$$(F^\times; \mathcal{A}) \models \text{ACF}_p^\times \text{ and } (F^\times; <, \mathcal{P}) \models T_{m,p}.$$

We observe that a similar construction as in the preceding paragraph allow us to obtain  $(M'; \mathcal{A}, <, \mathcal{P})$   $L_c^\times$ -extending  $(M; \mathcal{A}, <, \mathcal{P})$  such that  $(M'; \mathcal{A}) \models \text{ACF}_p^\times(\forall)$  and  $(M'; <, \mathcal{P}) \models T_{m,p}$ . We also note remind the reader that both  $\text{ACF}_p^\times$  and  $T_{m,p}$  have  $\forall\exists$ -axiomatization. Therefore, to obtain the desired  $(M'; \mathcal{A}, <, \mathcal{P})$ , we alternate the construction in the preceding paragraph and the construction in the observation and take union.

Let  $(F^\times; \mathcal{A}, <, \mathcal{P})$  be an  $L_c^\times$ -structure with  $(F^\times; \mathcal{A}) \models \text{ACF}_p^\times$ ,  $(F^\times; <, \mathcal{P}) \models T_{m,p}$ , and  $L_r^\times$ -definable  $X \subseteq (F^\times)^m$ ,  $L_m$ -definable  $Y \subseteq (F^\times)^m$  respectively permits multiplicative independence in  $(F^\times; \mathcal{A})$  and  $(F^\times; <, \mathcal{P})$ . We construct an  $L_c^\times$ -extension  $((F')^\times; \mathcal{A}, <, \mathcal{P})$  of  $(F^\times; \mathcal{A}, <, \mathcal{P})$  such that

$$((F')^\times; \mathcal{A}) \models \text{ACF}_p^\times, ((F')^\times; <, \mathcal{P}) \models T_{m,p} \text{ and } X((F')^\times) \cap Y((F')^\times) \neq \emptyset.$$

By given conditions of  $X$  and  $Y$ , we can find  $(N_1; \mathcal{A})$  elementarily extending  $(M; \mathcal{A})$  and  $(N_2; <, \mathcal{P})$  elementarily extending  $(M; <, \mathcal{P})$  with  $a' \in X(N_1)$  and  $b' \in X(N_2)$  such that  $a', b'$  are multiplicatively independent over  $M$ . Then there is a unique multiplicative  $L_g$ -isomorphism

$$\iota : \langle M, a' \rangle \rightarrow \langle M, b' \rangle$$

fixing  $M$  and mapping  $a'$  to  $b'$ . We equip  $\langle M, a' \rangle$  with  $<$  and  $\mathcal{P}$  by pullback. Hence,

$$(\langle M, a' \rangle; \mathcal{A}) \models \text{ACF}_p^\times(\forall) \text{ and } (M \langle a' \rangle; <, \mathcal{P}) \models T_{m,p}(\forall).$$

Then  $(\langle M, a' \rangle; \mathcal{A}, <, \mathcal{P})$  can be  $L_c^\times$ -embedded into  $((F')^\times; \mathcal{A}, <, \mathcal{P})$  with

$$((F')^\times; \mathcal{A}) \models \text{ACF}_p^\times \text{ and } ((F')^\times; <, \mathcal{P}) \models T_{m,p}$$

by the preceding paragraph. We note that  $a'$  is in  $X((F')^\times)$  because  $\text{ACF}_p^\times$  has quantifier elimination and  $a'$  is in  $Y((F')^\times)$  because  $T_{m,p}$  has quantifier elimination. Therefore  $X((F')^\times) \cap Y((F')^\times) \neq \emptyset$  as desired.

The main statement of the lemma follows from an application of the construction of the second paragraph, then repeated applications of the construction of the preceding paragraph for suitable choices of  $X, Y$  and taking union.  $\square$



*Proof of Theorem 1.3, Part 2.* We show that an arbitrary model of  $\text{ACFO}^-$  can be  $L_c$ -embedded into a model of  $\text{ACFO}$ . Suppose  $(F; <, \mathcal{P})$  is a model of  $\text{ACFO}_p^-$ . Then for some  $p$ ,

$$(F^\times; \mathcal{A}) \models \text{ACF}_p^\times \text{ and } (F^\times; <, \mathcal{P}) \models T_{m,p}(\forall).$$

By the preceding lemma,  $(F^\times; \mathcal{A}, <, \mathcal{P})$  can be  $L_c^\times$ -embedded into a model  $((F')^\times; \mathcal{A}, <, \mathcal{P})$  of  $\text{ACFO}_p^\times$ . By replacing 0 of  $F'$  if necessary, we can arrange that  $(F; <, \mathcal{P})$  is  $L_c$ -embeddable into  $(F'; <, \mathcal{P}) \models \text{ACFO}_p$ .  $\square$

Toward proving that  $\text{ACFO}$  is the model completeness of  $\text{ACFO}^-$ , the last component is showing that  $\text{ACFO}^-$  has the amalgamation property.

**Lemma 4.11.** *Let  $F \models \text{ACF}$  be  $\kappa^+$ -saturated and  $a, b, c$  be tuples of element in  $F$  of size  $< \kappa$ . If  $a$  is algebraically independent over  $b$ , then there is  $a'$  algebraically independent over  $(b, c)$  such that  $\text{tp}_r(a', b) = \text{tp}_r(a, b)$ .*

*Proof.* Suppose  $F, a, b, c$  are as give. As  $\text{ACF}$  is stable, we can find  $a'$  such that  $\text{tp}_r(a' \mid b, c)$  is the non-forking extension of  $\text{tp}_r(a \mid b)$ . By characterization of forking in  $\text{ACF}$ ,  $\text{trdeg}(a \mid b, c) = \text{trdeg}(a \mid b)$ . The conclusion thus follows.  $\square$

**Lemma 4.12.** *Suppose  $(M; <, \mathcal{P}) \models T_{m,p}$ ,  $a \in M$  and  $b$  is a tuple of elements in  $M$ . Then  $a \in \text{acl}_m(b)$  if and only if  $a$  is multiplicatively independent over  $b$ .*

*Proof.* For the forward direction, let  $(M; <, \mathcal{P}), a, b$  be as given and  $a \in \text{acl}_m(b)$ . We can arrange that  $b$  is a multiplicatively independent tuple of elements of length  $n$ . By assumption, there is  $\varphi \in L_m(x, y)$  such that the set defined by  $\varphi(x, b)$  has finitely many elements. We moreover arrange that for all  $b' \in M^n$ , the set defined by  $\varphi(x, b')$  has finitely many elements. Hence, the set defined by  $\varphi(x, y)$  in  $(\mathbb{U}_p; <, \mathcal{P})$  viewed as a  $L_m$ -substructure of  $(M; <, \mathcal{P})$  does not contain a  $q$ -hyper-arc. By the Corollary 4.7,  $(a, b)$  is not multiplicatively independent over  $\mathbb{U}_p$ . As  $b$  is multiplicative independent,  $a$  is multiplicatively dependent over  $b$ . The backward direction is clear.  $\square$

**Lemma 4.13.** *Let  $(M; <, \mathcal{P}) \models T_{m,p}$  be  $\kappa^+$ -saturated and  $a, b, c$  be tuples of element in  $F$  of size  $< \kappa$ . If  $a$  is multiplicatively independent over  $b$ , then there is  $a'$  multiplicatively independent over  $(b, c)$  such that  $\text{tp}_m(a', b) = \text{tp}_m(a, b)$ .*

*Proof.* Suppose  $(M; <, \mathcal{P}), a, b, c$  are as given. We can arrange that  $a$  is a tuple indexed by an ordinal  $\alpha$ . For  $\alpha = 0$ , there is nothing to prove so we first consider the case  $\alpha = 1$ . By the preceding lemma  $a \notin \text{acl}_m(b)$ . By compactness and  $\kappa$ -saturatedness of  $M$ , there is  $a'$  such that  $\text{tp}_m(a', b) = \text{tp}_m(a, b)$  and  $a'$  is not algebraic over  $(b, c)$ .

Suppose we have proven the statement for all  $\beta < \alpha$ . The case where  $\alpha$  is a limit ordinal is immediate, so suppose  $\alpha = \beta + 1$ . Then  $a = (a_{<\beta}, a_\beta)$ . By induction hypothesis, there is  $a'_{<\beta}$  such that  $a'_{<\beta}$  is multiplicatively independent over  $(b, c)$  and

$$\text{tp}_m(a_{<\beta}, b) = \text{tp}_m(a'_{<\beta}, b).$$

As  $(M; <, \mathcal{P})$  is a monster model, we can find  $a'_\beta \in M$  such that

$$\text{tp}_m(a', b) = \text{tp}_m(a, b) \text{ with } a' = (a'_{<\beta}, a'_\beta).$$

We can arrange to have  $a'_\beta$  multiplicatively independent over  $(a'_{<\beta}, b, c)$  by preceding paragraph. The conclusion follows.  $\square$

A theory  $T$  in a language  $L$  has the **disjoint amalgamation property** if for all  $M, M_1, M_2 \models T$  such that  $M_1, M_2$  extends  $M$ , we can find  $M' \models T$  and  $L$ -embeddings  $\iota : M \rightarrow M'$ ,  $\iota_1 : M_1 \rightarrow M'$  and  $\iota_2 : M_2 \rightarrow M'$  such that  $\iota_1, \iota_2$  extends  $\iota$  and  $\iota_1(M_1) \cap \iota_2(M_2) = \iota(M)$ .

**Proposition 4.14.**  $\text{ACFO}^\perp$  has the disjoint amalgamation property.

*Proof.* Throughout this proof, suppose  $(F_1; <, \mathcal{P}), (F_2; <, \mathcal{P})$  are models of  $\text{ACFO}^\perp$  extending  $(F; <, \mathcal{P}) \models \text{ACFO}^\perp$  as  $L_c$ -structure. We need to construct  $(F'; <, \mathcal{P}) \models \text{ACFO}^\perp$  and  $L_c$ -embeddings  $\iota, \iota_1$  and  $\iota_2$  as in the above definition.

As ACF has quantifier elimination and hence has amalgamation property, there is an algebraically closed field  $K$  extending  $F$  and  $L_r$ -embeddings  $f : F \rightarrow K$ ,  $f_1 : F_1 \rightarrow K$  and  $f_2 : F_2 \rightarrow K$  such that  $f_1, f_2$  extend  $f$ . We will replace  $K, f_1$  and  $f_2$  if necessary to also have that

$$f_1(F_1) \cap f_2(F_2) = f(F), \text{ and so } f_1(F_1^\times) \cap f_2(F_2^\times) = f(F^\times).$$

We can arrange that  $K$  is sufficiently saturated and  $f, f_1, f_2$  are identity map. Let  $a$  be an transcendence basis of  $F_1$  over  $F$ . By Lemma 4.11, we can find  $a'$  algebraically independent over  $F_2$  realizing the same type as  $a$  over  $F$ . Let  $F'_1$  be the algebraic closure in the field sense of  $a'$ . Then there is an  $L_r$ -isomorphism  $r : F_1 \rightarrow F'_1$  which is identity on  $F$ . If  $a'$  is a finite tuple, we have  $F'_1 \cap F_2 = F$  because algebraic independence satisfies exchange properties. In general, we still have  $F'_1 \cap F_2 = F$  as  $F'_1$  is the union of the algebraic closures of finite sub-tuples of  $a'$ . Replace  $\text{id} : F_1 \rightarrow K$  by  $\text{id} \circ r : F_1 \rightarrow K$  we achieve  $f_1(F_1) \cap f_2(F_2) = f(F)$ .

We also have that there is  $N \models T_m$  extending  $(F^\times; <, \mathcal{P})$  as  $L_m$ -structure and  $L_m$ -embedding  $g : F^\times \rightarrow N$ ,  $g_1 : F_1^\times \rightarrow N$  and  $g_2 : F_2^\times \rightarrow N$  such that  $g_1, g_2$  extends  $g$  and

$$g_1(F_1^\times) \cap g_2(F_2^\times) = g(F^\times).$$

The proof is the same as the above proof but use multiplicative independence instead of algebraic independence and Lemma 4.13 instead of 4.11.

Keeping the notations as in the preceding two paragraph, we have that

$$\langle f_1(F_1^\times), f_2(F_2^\times) \rangle \cong_{L_g} \langle g_1(F_1^\times), g_2(F_2^\times) \rangle$$

where the former is the subgroup of  $K^\times$  generated by  $f_1(F_1^\times), f_2(F_2^\times)$  and the latter is the subgroup of  $N$  generated by  $g_1(F_1^\times), g_2(F_2^\times)$ . This allows us to equip an  $L_c^\times$ -structure on  $M = \langle f_1(F_1^\times), f_2(F_2^\times) \rangle$  with  $\mathcal{A}$  defined by restriction from the  $L_r^\times$ -reduct  $(K^\times; \mathcal{A})$  of  $K$  and  $<, \mathcal{P}$  defined by pullback. By construction,

$$(M; \mathcal{A}) \models \text{ACF}_p^\times(\forall) \text{ and } (M; <, \mathcal{P}) \models T_{m,p}(\forall).$$

Therefore, by 4.10, there is a reduct  $((F')^\times; \mathcal{A}, <, \mathcal{P})$  of  $(F'; <, \mathcal{P}) \models \text{ACFO}$  and an  $L_c^\times$ -embedding  $j^\times$  from  $(M; \mathcal{A}, <, \mathcal{P})$  into a  $((F')^\times; \mathcal{A}, <, \mathcal{P})$ . Set

$$\iota^\times = j^\times \circ (f \upharpoonright F^\times), \iota_1^\times = j^\times \circ (f_1 \upharpoonright F_1^\times) \text{ and } \iota_2^\times = j^\times \circ (f_2 \upharpoonright F_2^\times).$$

Clearl,  $\iota_1^\times, \iota_2^\times$  extends  $\iota^\times$  and  $\iota_1^\times(F_1^\times) \cap \iota_2^\times(F_2^\times) = \iota^\times(F^\times)$ . We can easily extends them to obtain  $\iota, \iota_1, \iota_2$  with the desired properties.  $\square$

*Proof of Theorem 1.3, part 3.* By a standard test (see Exercise 3.4.14 of [Mar02]), the preceding proposition together with part 1 and 2 of the proof implies that  $\text{ACFO}$  is the model completion of  $\text{ACFO}^\perp$ .  $\square$

For the rest of the section,  $z = (z_1, \dots, z_k)$  is a  $k$ -tuple of variables with  $k > 0$  and  $t$  is a single variable. We deduce a description of definable sets in a model of ACFO which is slightly more precise than the version in Theorem 1.3. A formula  $\varphi(x) \in L_c(x)$  is a **special formula** associated to  $P \in \mathbb{Z}[z, t]$  if it has the form

$$\exists z (z_1 < \dots < z_k \wedge \rho_P(x, z) \wedge \varphi_r(x, z) \wedge \varphi_m(x, z)).$$

where  $\varphi_r \in L_r(x, z)$ ,  $\varphi_m \in L_m(x, z)$  and  $\rho_P(x, z)$  is a formula  $L_r(x, z)$  such that for all  $a \in F^m$  and  $c \in F^k$ ,  $\rho_P(a, c)$  if and only if  $c_1, \dots, c_k$  are the only non-zero roots of  $P(t, a)$  in  $F$ . The latter has a first-order expression as we can define the multiplicity of a root of  $P$  using derivations of  $P$ . We do allow  $m = 0$ , in which case we call the special formula a **special statement**.

**Lemma 4.15.** *Suppose  $(F; <, \mathcal{P})$  and  $(F'; <, \mathcal{P})$  are models of  $\text{ACFO}^\perp$ ,  $a \in F^m$ ,  $a' \in (F')^m$  and  $(F; <, \mathcal{P}, a) \models \varphi(a) \Rightarrow (F'; <, \mathcal{P}, a') \models \varphi(a')$  for all special formulas  $\varphi \in L_c(x)$ . Then there is an  $L_c$ -isomorphism from  $(\text{acl}_r(a); <, \mathcal{P})$  to  $(\text{acl}_r(a'); <, \mathcal{P})$  with  $a \mapsto a'$ .*

*Proof.* Suppose the notations are as given. It is easy to see that  $F$  and  $F'$  have the same characteristic. We can easily choose a sequence  $(P_l)_{l>0}$  of polynomials in  $\mathbb{Z}[t, x]$  with  $|t| = 1$  such that with  $Z_{l,F}$  the set of non-zero roots of  $P_l(t, a)$  in  $F$  for  $l > 0$ , we have that

$$\bigcup_{l>0} Z_{l,F} = \text{acl}_r(a) \text{ and } Z_{l,F} \subseteq Z_{l+1,F} \text{ for all } l > 0.$$

Define  $Z_{l,F'}$  similarly for  $l > 0$ . From the hypothesis, for each  $l > 0$ , there is a map  $f_l : Z_{l,F} \rightarrow Z_{l,F'}$  such that  $f_l$  preserve the restrictions of  $<, \mathcal{P}$  and the restrictions of the graph of addition and multiplication. For  $l' \in \mathbb{N}^{\geq l}$ , we have that  $f_{l'}$  extends  $f_l$  where  $f_{l'}$  is constructed similarly. Thus,  $f = \bigcup_{l>0} f_l$  is an  $L_c$ -embedding. This is an isomorphism as the image is algebraically closed.  $\square$

**Proposition 4.16.** *Every formula in  $L_c(x)$  is equivalent to a disjunction of special formulas in  $L_c(x)$  across all models of ACFO.*

*Proof.* We note that if  $(F; <, \mathcal{P}) \models \text{ACFO}$  and  $a \in F^m$ , then  $(\text{acl}_r(a); <, \mathcal{P}) \models \text{ACFO}^\perp$ . By a standard test, the the preceding lemma and previous parts of theorem 1.3, every formula in  $L_c(x)$  is equivalent to a positive boolean combination of special formulas in  $L_c(x)$  across all models of ACFO. The conclusion follows the observation that if  $\varphi_P$  and  $\varphi_Q$  are special formulas associated to  $P$  and  $Q$ , then  $\varphi_P \wedge \varphi_Q$  is equivalent to a disjunction of special formulas associated to  $PQ$ .  $\square$

**Lemma 4.17.** *Sets defined by positive boolean combinations of special formulas in a model of ACFO are one-to-one projections of quantifier-free definable sets.*

*Proof.* As ACF and  $T_{m,p}$  admit quantifier elimination, a special formula defines a one-to-one projection quantifier-free definable sets. The collection of the latter is closed under finite unions. The conclusion follows.  $\square$

*Proof of Theorem 1.3, part 4.* It follows from the preceding proposition and lemma, definable sets in an ACFO models are one-to-one projections of quantifier-free definable sets.  $\square$

*Proof of Corollary 1.4.* The forward direction follows by applying Lemma 4.15 when  $a$  is the empty tuple. The backward direction follows from Theorem 1.3 noting that if  $(F; <, \mathcal{P}) \models \text{ACFO}$  then  $(\text{Abs}(F); <, \mathcal{P}) \models \text{ACFO}^\perp$ .  $\square$

The above leads to trying to understand  $(\text{Abs}(F); <, \mathcal{P})$  in  $(F; <, \mathcal{P}) \models \text{ACFO}$ . There are some good answers in the case where  $\text{char}(F)$  is prime.

*Proof of Proposition 1.5.* Suppose  $(F; <, \mathcal{P}) \models \text{ACFO}$  has  $\text{char}(F) = p \neq 0$ . In this case  $\text{Abs}(F) = \text{acl}_r(\mathbb{F}_p)$  and hence  $\text{Abs}^\times(F) = \text{acl}_m(\emptyset)$  in  $F^\times$ . As the  $T_{a,p}$ -model  $(\mathbb{Z}_{(p)}; <, \mathcal{D}, \pm 1)$  is uniquely  $L_a$ -embeddable into any model of  $T_{a,p}$  and  $\mathcal{F}_m$  is functorial, there is a map  $\delta : \mathbb{U}_{(p)} \rightarrow F^\times$  which is an  $L_m$ -embedding of  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  into  $(F^\times; <, \mathcal{P})$ . The image  $\delta(\mathbb{U}_{(p)})$  is precisely  $\text{acl}_m(\emptyset)$ . Therefore  $(\text{Abs}^\times(F); <, \mathcal{P})$  is  $L_m$ -isomorphic to  $(\mathbb{U}_{(p)}; <, \mathcal{P})$  via the isomorphism  $\delta^{-1} : \text{Abs}(F) \rightarrow \mathbb{U}_{(p)}$ . This is the same as saying  $(\text{Abs}(F); <, \mathcal{P})$  is the standard model correspond to  $\text{Abs}(F)$  and the character map  $\iota \circ \delta^{-1}$  where  $\iota$  is the canonical embedding of  $\mathbb{U}_{(p)}$  into  $\mathbb{C}$ . The remaining part of the statement follows from Theorem 1.3.  $\square$

Let  $p$  be prime. For each  $n > 0$ , let  $\Phi_{p,n} \in \mathbb{Z}[x]$  with  $|x| = 1$  be the  $p^n - 1$ -th cyclotomic polynomial. Viewed as an element of  $\mathbb{F}_p[x]$  with  $|x| = 1$  in an obvious way,  $\Phi_{p,n}$  factors into  $\varphi(p^n - 1)/n$  monic irreducible polynomials, each of degree  $n$ , in  $\mathbb{F}[x]$  where  $\phi$  is the Euler totient. We will call each of the irreducible component an  $(p, n)$ -**cyclotomic factor**. Suppose  $\Psi = (\Psi_n)_{n>0}$  with  $\Psi_n$  a  $(p, n)$ -cyclotomic factor for  $n > 0$ . We say that  $\Psi$  is a **coherent** sequence of  $(p, n)$ -cyclotomic factors if for all  $n, n' \in \mathbb{N}$  with  $0 < n < n'$ , for all roots  $a$  of  $\Psi_n$  in  $\mathbb{F}_{p^n}$ , there is a root  $a'$  of  $\Psi_{n'}$  in  $\mathbb{F}_{p^{n'}}$  such that with

$$a = (a')^k \text{ where } k = \frac{p^{n'} - 1}{p^n - 1}.$$

We denote by  $\text{Coh}_p$  the set of all coherent sequences of  $(p, n)$ -cyclotomic factor.

Suppose  $F$  is the algebraic closure of  $\mathbb{F}_p$ ,  $\chi : F^\times \rightarrow \mathbb{C}$  is an injective multiplicative preserving map and  $(F; <, \mathcal{P})$  is the associated standard model. Let  $a_n$  be the smallest  $p^n - 1$ -th root of unity with respect to  $<$  and  $\Psi_n$  the monic minimal polynomial of  $a_n$ . Then  $\Psi_{F,\chi} = (\Psi_n)_{n>0}$  is a coherent sequence of  $(p, n)$ -cyclotomic factors. If  $(F'; <, \mathcal{P})$  is a standard model given by  $\chi' : (F')^\times \rightarrow \mathbb{C}$  and is isomorphic to  $(F; <, \mathcal{P})$ , then the similarly defined  $\Psi_{F',\chi'}$  is the same as  $\Psi_{F,\chi}$ . Let  $\text{Eleq}_p$  be the collection of isomorphism classes of standard models  $(F; <, \mathcal{P})$ . We note that this is also the collection of elementary equivalence classes of  $\text{ACFO}_p$  by Corollary 1.4. The above define a map  $\text{Inv} : \text{Eleq}_p \rightarrow \text{Coh}_p$ . We have that:

**Proposition 4.18.** *The map  $\text{Inv} : \text{Eleq}_p \rightarrow \text{Coh}_p$  is a bijection.*

*Proof.* Suppose  $\Psi = (\Psi_n)_{n>0}$  is a coherent sequence of  $(p, n)$ -cyclotomic factors. By König's lemma we can choose a sequence  $(a_n)_{n>0}$  such that for all  $n > 0$ ,  $a_n$  is a solution of  $\Psi_n$  in  $\mathbb{F}_{p^n}$  of  $a_n$  and the relationship between  $a_n$  and  $a_{n'}$  is as specified in the above equation. Let  $F$  be the algebraic closure of  $\mathbb{F}_p$ , define  $\chi : F^\times \rightarrow \mathbb{C}$  by mapping  $a_n$  to the smallest  $(p^n - 1)$ -th root of unity in  $\mathbb{T}$  with respect to  $<$  on  $\mathbb{T}$ . We can check that  $\Psi_{F',\chi'} = \Psi_{F,\chi}$ .

Suppose  $(F; <, \mathcal{P})$  and  $(F'; <, \mathcal{P})$  has  $\Psi_{F',\chi'} = \Psi_{F,\chi}$ . Let  $(a_n)_{n>0}$  be the sequence of smallest  $p^n - 1$ -th root in  $(F; <, \mathcal{P})$  and  $(a'_n)_{n>0}$  defined likewise for  $(F'; <, \mathcal{P})$ . Then as  $\Psi_{F',\chi'} = \Psi_{F,\chi}$ , there is a unique field isomorphism  $\iota : F \rightarrow F'$  mapping  $a_n \rightarrow a'_n$  for  $n > 0$ . We can check that under  $\iota$ ,  $\chi$  maps to  $\chi'$  and therefore  $(F; <, \mathcal{P})$  and  $(F'; <, \mathcal{P})$  are isomorphic.  $\square$

We understand the case where  $p$  is zero much less. However, we still have:

**Proposition 4.19.** *If  $(F; <, \mathcal{P}) \models \text{ACFO}_0$  then  $(\text{Abs}(F); <, \mathcal{P}) \models \text{ACFO}_0^-$ .*

*Proof.* Suppose  $(F; <, \mathcal{P})$  is as given. As every model of  $\text{ACFO}^-$  is embeddable into a model of  $\text{ACFO}$  by Theorem 1.3, we can arrange that  $(F; <, \mathcal{P}) \models \text{ACFO}$ . The case where  $\text{char}(F) = p$  is covered in the preceding proof, so we assume  $\text{char}(F) = 0$ . Let  $(G; <, \mathcal{D}, \pm 1)$  be the  $L_a$ -cover of  $(F^\times; <, \mathcal{P})$ . It suffices to check that  $(G; <, \mathcal{D}, \pm 1) \models T_{a,0}$  or in other words, all  $(k, a)$  with  $a \in F^\times$  is  $n$ -divisible in  $G$ . We can arrange that  $0 \leq k < n$ . The equation  $x^n = a$  has exactly  $n$  solutions  $c_1, \dots, c_n$ . Therefore, for some  $i \in \{1, \dots, n\}$ , we must have  $n \cdot (0, c_i) = (k, a)$ . The conclusion follows.  $\square$

We next prove decidability results about  $\text{ACFO}$  and  $\text{ACFO}_p$  for arbitrary  $p$ .

**Lemma 4.20.** *The classes  $\text{ACFO}$  and  $\text{ACFO}_p$  for an arbitrary  $p$  have recursively axiomatization.*

*Proof.* We first prove that the set of statements true in all models of  $T_a$  is recursive. For arbitrary  $p$ ,  $T_{a,p}$  is recursively axiomatizable and complete and hence the set of statements true in all models of  $T_{a,p}$  is recursive. The set of statements true in all models of  $T_a$  is recursively enumerable as  $T_a$  has a recursive axiomatization. On the other hand, a statement is not true in all models of  $T_a$  if and only if it is not true in some model of  $T_{a,p}$ . Hence, this set is also recursive enumerable as well. Thus, the set of statements true in all models of  $T_a$  is recursive.

We prove the main statement of the lemma. Since every model of  $T_m$  is interpretable in a model of  $T_a$ , therefore the set of statements true in all models of  $T_m$  is also recursive. It is well known that  $\text{ACF}$  has a recursive axiomatization. Therefore  $\text{ACFO}^-$  is recursively axiomatizable. The schema in part 2 of the proof of theorem 1.2 is also recursive axiomatizable. Thus,  $\text{ACFO}$  is recursively axiomatizable. It follows that  $\text{ACFO}_p$  also have recursive axiomatization.  $\square$

Let  $M$  be a multiplicative group. Suppose  $c$  is in  $M^k$ . Let  $S_c$  be the set of  $i \in \{1, \dots, k\}$  such that  $c_i$  is multiplicatively dependent over  $c_1, \dots, c_{i-1}$ . Again,  $z = (z_1, \dots, z_k)$ . For  $i \in S_c$ , let  $\varepsilon_i(z)$  be the equation

$$z_i^{l_i} \dots z_1^{l_1} = z_i^{l'_i} \dots x_1^{l'_1}$$

satisfied by  $a$  such that for all  $j \in \{1, \dots, i\}$ ,  $l_j, l'_j \geq 0$ , either  $l_j = 0$  or  $l'_j = 0$ , and  $l_i > 0$  is chosen to be smallest possible. Set  $\theta \in L_g(z)$  to be the formula  $\bigwedge_{i \in S_c} \varepsilon_i(z)$ . We call the above  $\theta$  the **multiplicative dependence pattern** of  $c$ . We also denote by  $\theta$  the obvious interpretations of the above multiplicative dependence pattern in  $L_m, L_r^\times, L_r, L_c^\times$  and  $L_c$ . The following lemma is an immediate observation:

**Lemma 4.21.** *Suppose  $M, M'$  are multiplicative groups and  $c \in M^k, c' \in (M')^k$  are such that  $c$  and  $c'$  have the same multiplicative dependence pattern. Then*

$$\langle c \rangle \cong_{L_g} \langle c' \rangle$$

where the former is the subgroup of  $M$  generated by  $c$  and the later is defined for  $c'$  and  $M'$ .  $\square$

Suppose  $P \in \mathbb{Z}[t]$  is non-zero. Let  $c_1, \dots, c_k$  be all the non-zero roots of  $P$  in a field  $K$ . For a permutation  $\sigma$  in  $S_k$ , set  $c_\sigma = (c_{\sigma(1)}, \dots, c_{\sigma(k)})$ . Then define  $\Theta_P(K)$  to be the set of multiplicative dependence patterns of  $c_\sigma$  as  $\sigma$  ranges over  $S_k$ . Clearly,  $\Theta_P(K) = \Theta_P(K')$  for fields  $K, K'$  such that  $K \equiv_{L_r} K'$ . Define  $\Theta_P(\text{ACF}_p)$  to be  $\Theta_P(F)$  for  $F \models \text{ACF}_p$ .

**Lemma 4.22.** *There is an algorithm which compute for given  $p$  and given non-zero  $P \in \mathbb{Z}[t]$  the set  $\Theta_P(\text{ACF}_p)$ .*

*Proof.* We first show for fixed prime  $p$  that there is an algorithm which compute for given non-zero  $P \in \mathbb{Z}[t]$  the set  $\Theta_P(\text{ACF}_p)$ . Suppose  $P$  has degree  $k \geq 0$ . We observe that  $\Theta_P(\text{ACF}_p) = \Theta_P(\mathbb{F}_q)$  where  $q = p^{k!}$ . Choose a  $\xi$  be a primitive root of unity in  $\mathbb{F}_q^\times$ . We note that any non-zero root of  $P$  in  $\mathbb{F}_q$  can be written as power of  $\xi$ . From here we can find  $\Theta_P(\mathbb{F}_q)$ . It is easy to see that all the above steps can be carried out algorithmically.

It remains to show that there is an algorithm which compute for given non-zero  $P \in \mathbb{Z}[t]$  the set  $\Theta_P(\text{ACF}_0)$ . We will first describe a pseudo algorithm and then argue that the steps of this can be done algorithmically. Find the splitting field  $K$  of  $P$  over  $\mathbb{Q}$ . Let  $c_1, \dots, c_k$  be the non-zero roots of  $P$  in  $K$ . It is easy to see that  $\Theta_P(\text{ACF}_0) = \Theta_P(K)$ . Moreover, computing  $\Theta_P(K)$  can be reduced to the problem of finding a set of generator for the additive group  $\{(l_1, \dots, l_k) \in \mathbb{Z}^k : c_1^{l_1} \dots c_k^{l_k} = 1\}$  through solving linear equations over  $\mathbb{Z}$ .

We consider an intermediate problem of finding a set of generators for the additive group

$$\{(l_1, \dots, l_k) \in \mathbb{Z}^k : c_1^{l_1} \dots c_k^{l_k} \text{ is a unit in } \mathcal{O}_K\}.$$

Find a finite set of prime ideals  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  of the ring of integers  $\mathcal{O}_K$  which consists of all the prime ideals in the factorizations of the fractional ideals  $(c_1), \dots, (c_m)$ . For  $i \in \{1, \dots, m\}$ , let  $v_i : K^\times \rightarrow \mathbb{Z}$  be the valuation associated to  $\mathfrak{p}_i$ . We obtain the desired set of generators by solving the system of  $m$  equations where the  $i$ -th equation is  $l_1 v_i(c_1) + \dots + l_k v_i(c_k) = 0$  for  $i \in \{1, \dots, m\}$ .

We note that the group  $\{(l_1, \dots, l_k) \in \mathbb{Z}^k : c_1^{l_1} \dots c_k^{l_k} = 1\}$  is a subgroup of the group  $\{(l_1, \dots, l_k) \in \mathbb{Z}^k : c_1^{l_1} \dots c_k^{l_k} \text{ is a unit in } \mathcal{O}_K\}$ . The later is isomorphic to  $\mathbb{Z}^{k'}$  with  $k' \geq 0$ . Hence, after a change of basis and using the preceding paragraph we can reduce to the following problem: for given  $d_1, \dots, d_{k'}$  in the unit group of  $\mathcal{O}_K$ , find a set of generators of the additive group

$$\{(l_1, \dots, l_{k'}) \in \mathbb{Z}^{k'} : (d_1)^{l_1} \dots (d_{k'})^{l_{k'}} = 1\}.$$

Choose  $u_1, \dots, u_n, u_{n+1}$  be a set of generator of the unit group of  $\mathcal{O}_K$  such that  $u_1, \dots, u_n$  are multiplicative independent and  $u_{n+1}$  generates the group of roots of unity in  $K$ . Suppose for  $i \in \{1, \dots, k'\}$ , we have  $d_i = u_1^{w_1(d_i)} u_{n+1}^{w_{n+1}(d_i)} \dots u_{n+1}^{w_{n+1}(d_i)}$  with  $w_1(d_i), \dots, w_{n+1}(d_i) \in \mathbb{Z}$ . We obtain the desired set of generators by solving the system of  $n+1$  equations where the  $i$ -th equation is  $l_1 w_i(d_1) + \dots + l_{k'} w_i(d_{k'}) = 0$  and the  $(n+1)$ -th equation is  $l_1 w_{n+1}(d_1) + \dots + l_{k'} w_{n+1}(d_{k'}) \equiv 0 \pmod{h}$  where  $h$  is the order of  $u_{n+1}$ .

We note that the all the above steps can be done algorithmically. The non-trivial steps include: finding  $K$ , finding  $\mathcal{O}_K$  in  $K$ , finding  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  in  $K$ , finding  $v_i(c_j)$  for  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, k\}$  as in the third paragraph, and finding  $u_1, \dots, u_n, u_{n+1}$  as in the forth paragraph. These are standard results of computational algebraic number theory which can be found in [Coh93, 4.2, 4.6, 4.8, 6.5]  $\square$

**Lemma 4.23.** *Suppose  $\exists z (z_1 < \dots < z_k \wedge \rho_P(z) \wedge \varphi_r(z) \wedge \varphi_m(z))$  is a special statement associated to  $P \in \mathbb{Z}[t]$  and  $\theta \in L_g(z)$  is a multiplicative dependence pattern. Then for all  $p$ , the following are equivalent:*

- (1) *The statement  $\exists z (\theta(z) \wedge \rho_P(z) \wedge \varphi_r(z))$  holds in some model (all models) of  $\text{ACF}_p$  and the statement  $\exists z (\theta(z) \wedge (z_1 < \dots < z_k) \wedge \varphi_m(z))$  holds in some model (all models) of  $T_{m,p}$ .*
- (2) *The statement  $\exists z (\theta(z) \wedge (z_1 < \dots < z_k) \wedge \rho_P(z) \wedge \varphi_r(z) \wedge \varphi_m(z))$ , holds in some model of  $\text{ACFO}_p$ .*

*Proof.* We will only prove (1) implies (2) as the other direction is clear. Suppose  $F \models \text{ACF}_p$ ,  $c \in F^k$  satisfies  $\theta(z) \wedge \rho_P(z) \wedge \varphi_r(z)$ ,  $(M; <, \mathcal{P}) \models T_{m,p}$  and  $d \in M^k$  satisfies  $\theta(z) \wedge (z_1 < \dots < z_k) \wedge \varphi_m(z)$ . Then, by Lemma 4.21, there is an  $L_g$ -isomorphism  $\iota : \langle c \rangle \rightarrow \langle d \rangle$ . We can equip  $\langle c \rangle$  with an  $L_c^\times$ -structure with  $\mathcal{A}$  defined by restriction from the  $L_r^\times$ -reduct  $(F^\times; \mathcal{A})$  of  $F$  and  $<, \mathcal{P}$  defined by pullback. Then

$$(\langle c \rangle; \mathcal{A}) \models \text{ACF}_p^\times(\forall) \text{ and } (\langle c \rangle; <, \mathcal{P}) \models T_{m,p}(\forall).$$

By Proposition 4.10,  $(\langle c \rangle; \mathcal{A}, <, \mathcal{P})$  can be embedded as an  $L_c^\times$ -structure into  $((F')^\times; \mathcal{A}, <, \mathcal{P})$ . As  $\text{ACF}_p^\times$  and  $T_{m,p}$  have quantifier elimination,  $c$  satisfies

$$\theta(z) \wedge (z_1 < \dots < z_k) \wedge \rho_P(z) \wedge \varphi_r(z) \wedge \varphi_m(z).$$

The conclusion follows.  $\square$

*Proof of Proposition 1.7.* It suffices to show that the set of all  $L_c$ -statements which hold in all models of ACFO is recursive. By Lemma 4.20, ACFO has a recursive axiomatization, so the set  $\{\sigma \in L_c : \text{ACFO}_p \models \sigma\}$  is recursively enumerable. Hence, it remains to showing that the set

$$\{\sigma \in L_c : (F; <, \mathcal{P}) \models \neg\sigma \text{ for some } (F; <, \mathcal{P}) \models \text{ACFO}_p\}$$

is also recursively enumerable. This reduces to the problem of finding an algorithm to decide for a given  $p$  and a given  $L_c$ -statement  $\neg\sigma$  whether  $\neg\sigma$  can hold in some model of  $\text{ACFO}_p$ . By Lemma 4.16, we can arrange that the  $\neg\sigma$  is a special formula  $\exists z (z_1 < \dots < z_k \wedge \rho_P(z) \wedge \varphi_r(z) \wedge \varphi_m(z))$  associated to  $P \in \mathbb{Z}[t]$  as in the preceding lemma. By Lemma 4.22, this reduces to finding an algorithm deciding whether

$$\exists z (\theta_c(z) \wedge (z_1 < \dots < z_k) \wedge \rho_P(z) \wedge \varphi_r(z) \wedge \varphi_m(z))$$

holds in some model of  $\text{ACFO}_p$ . Such an algorithm exists by the preceding lemma and the decidability of  $\text{ACF}_p$  and  $T_{m,p}$ .  $\square$

*Proof of Proposition 1.8.* We first show that  $\text{acl}_c$  and  $\text{acl}_r$  coincide. Let  $(F_1; <, \mathcal{P})$  be a model of ACFO and  $A \subseteq F_1$ . It is clear that  $\text{acl}_r(A) \subseteq \text{acl}_c(A)$ . Suppose  $a \in F_1$  is in  $\text{acl}_c(A)$ . Then there is  $\varphi(x) \in L_{c,A}(x)$  such that  $\varphi(a)$  and  $\varphi(x)$  defines a set with  $k$  elements in  $F_1$  with  $k > 0$ . Let  $F$  be the field with the underlying set  $\text{acl}_c(A)$ ; hence  $(F; <, \mathcal{P})$  is a model of  $\text{ACFO}^-$ . By Proposition 4.14 and Theorem 1.3, we can arrange that there are ACFO-models  $(F_2; <, \mathcal{P})$  and  $(F'; <, \mathcal{P})$  such that both  $(F_1; <, \mathcal{P})$  and  $(F_2; <, \mathcal{P})$  are ACFO-submodels of  $(F'; <, \mathcal{P})$  and  $F_1 \cap F_2 = F$ . By the preceding theorem,  $\varphi(x)$  also define a set with  $k$ -elements in  $F_2$  and in  $F'$ . This implies all the elements defined by  $\varphi(x)$  must be in  $F$ . The fact that  $\text{dcl}_c$  coincides with  $\text{acl}_c$  simply follows from the fact that in  $(F; <, \mathcal{P}) \models \text{ACFO}$ ,  $<$  is a total ordering on  $F^\times$ .  $\square$

## 5. COMBINATORIAL TAMENESS

We keep the notational conventions in the first paragraphs of the preceding three sections. Moreover,  $n'$  ranges over the set of natural numbers and  $y' = (y'_1, \dots, y'_{n'})$ . Let  $\kappa, \lambda, \mu, \nu$  be cardinals. A  $\kappa$ -sequence with terms in  $X^\nu$  is officially an element  $a \in X^{\kappa \times \nu}$ . However, we also view  $a$  in an obvious way as a function mapping  $i \in \kappa$  to  $a_i \in X^\nu$ , which we will call the  $i$ -th term of  $a$ . Likewise, a  $(\kappa, \lambda)$ -array with terms in  $X^\nu$  is an element  $a \in M^{\kappa \times \lambda \times \nu}$ . We also think of  $a$  as a function mapping  $i \in \kappa$  to the  $\lambda$ -sequence  $a_i \in M^{\lambda \times \nu}$  and mapping  $(i, j) \in \kappa \times \lambda$  to  $a_{ij} \in X^\nu$ . We call the above  $a_i$  the  $i$ -th row of  $a$ ; for  $(i, j) \in \kappa \times \lambda$  and  $a_{ij}$  the  $(i, j)$ -term of  $a$ . We note that that our definitions allow us to talk in a natural way of the type of a sequence or an array. For  $a \in X^\omega$ , we let  $a \upharpoonright m \in X^m$  be the tuple consisting of the first  $m$  terms of  $a$  (but with index starting from 1). If  $a$  is a tuple of elements in  $F^\times$ , we let  $\text{acl}_r^\times(a) = \text{acl}_r(a) \cap F^\times$ . It is easy to see that  $\text{acl}_r^\times$  is also the model theoretic algebraic closure of  $a$  in the language  $L_r^\times$ .

Let  $T$  be a theory in a language  $L$  and  $\varphi(x, y)$  be in  $L(x, y)$  with  $|x| = 1$  and  $|y| = n$ . We say  $\varphi(x, y)$  has **TP**<sub>2</sub> in  $T$  if for *some* infinite cardinals  $\kappa, \lambda$ , there is  $M \models T$  and  $b$  a  $(\kappa, \lambda)$ -array  $b$  with terms in  $M^n$  such that with  $X_{ij}$  the set defined by  $\varphi(x, b_{ij})$  we have:

- (1) for all  $i \in \kappa$  and distinct  $j, j' \in \lambda$ ,  $X_{ij} \cap X_{ij'}$  is empty;
- (2) for every choice of  $j : \kappa \rightarrow \lambda$ , we have  $\bigcap_{i \in \kappa} X_{ij(i)}$  is non-empty.

We call  $b$  as in the above definition a *witness* for the **TP**<sub>2</sub> of  $\varphi(x, y)$  in  $T$ ; here  $M$  is thought of as part of the data specified along with  $b$ . It follows immediately from compactness that  $\varphi(x, y)$  has **TP**<sub>2</sub> in  $T$  if and only if for *all* infinite  $\kappa, \lambda$ , there is a  $(\kappa, \lambda)$ -array  $b$  with term in  $M^n$  for some  $M \models T$  witnessing the **TP**<sub>2</sub> of  $\varphi(x, y)$  in  $T$ .

We say that  $T$  has **TP**<sub>2</sub> if for some  $n$  and some  $\varphi(x, y)$  in  $L(x, y)$  with  $|x| = 1$  and  $|y| = n$ ,  $\varphi(x, y)$  has **TP**<sub>2</sub> in  $T$ . Otherwise, we say  $T$  has **NTP**<sub>2</sub>. This definition can be seen equivalent to [Che14, Definition 3.1] by [Che14, Theorem 2.9] and [Che14, Lemma 3.2].

We note that **ACFO** and **ACFO**<sup>×</sup> are bi-interpretable. Hence, the former has **NTP**<sub>2</sub> if and only if the later does. The first step is to show that if **ACFO**<sup>×</sup> has **TP**<sub>2</sub> then it has a **TP**<sub>2</sub>-witness of a special type.

**Lemma 5.1.** *Suppose **ACFO**<sup>×</sup> has **TP**<sub>2</sub>. Then we can find a formula  $\varphi^\times(x, y, y')$  in  $L_c^\times(x, y, y')$  with  $|x| = 1$ ,  $|y| = n$  and  $|y'| = n'$ , a monster model  $(F^\times; \mathcal{A}, <, \mathcal{P})$  of **ACFO**<sup>×</sup>, a  $(\kappa, \lambda)$ -array  $b$  with terms in  $(F^\times)^\omega$ , and  $c \in (F^\times)^\omega$  such that:*

- (1) the array  $(b_{ij} \upharpoonright n, c \upharpoonright n')_{(i,j) \in \kappa \times \lambda}$  is a witness for the **TP**<sub>2</sub> of  $\varphi^\times(x, y, y')$ ;
- (2)  $c$  is a listing of elements in  $\text{acl}_r^\times(c)$ ;
- (3) for all  $(i, j) \in \kappa \times \lambda$ ,  $b_{ij}$  is a listing of elements in  $\text{acl}_r^\times(c, b_{ij} \upharpoonright n) \setminus \text{acl}_r^\times(c)$ ;
- (4) for all  $i \in \kappa$ , and distinct  $j, j' \in \lambda$ ,  $b_{ij}$  and  $b_{ij'}$  are relative algebraically independent over  $c$ , that is  $\text{trdeg}(b_{ij'} \mid c, b_{ij}) = \text{trdeg}(b_{ij'} \mid c)$ ;
- (5) the rows of  $b$  are mutually indiscernible over  $c$ ;
- (6)  $b$  as a sequence of its rows is indiscernible over  $c$ .



*Proof.* To avoid notation overflow, we will allow a symbol to evolve and potentially denote different type of objects at different parts of the proof. Suppose  $\text{ACFO}^\times$  has  $\text{TP}_2$  then there is  $\varphi^\times \in L_c^\times(x, y)$ ,  $F \models \text{ACFO}$  and an  $(\kappa, \lambda)$ -array  $b$  with terms in  $(F^\times)^n$  which witness the  $\text{TP}_2$  of  $\varphi^\times$ . We can arrange that  $F$  is a monster model and the rows of  $b$  are mutually indiscernible.

We next introduce  $c$  and modify  $b$  above to satisfy (4). By reducing the size of the array if necessary, we arrange that  $\kappa = \omega, \lambda = \omega$ . Let  $c \in (F^\times)^\omega$  be the concatenation of all the terms  $b_{ij}$  with  $j < n$  (that is the first  $n$  “columns” of the array). We can see that for all  $i$ , the sequence  $(b_{ij})_{j \geq n}$  is indiscernible over  $\{c\} \cup \{b_{i'j} : i' \neq i, j \geq n\}$ . Let  $k, l, l'$  be in  $\omega$  such that  $l < l'$  and  $l, l' \geq n$ . As the tuple  $b_{kl'}$  has length  $n$ , we can find  $m, m'$  with  $m + m' \leq n$ , tuples  $b_{i_1 j_1}, \dots, b_{i_m j_m}$  in  $\{b_{ij} : i \neq k, j < n\}$ , and tuples  $b_{k j'_1}, \dots, b_{k j'_{m'}}$  in  $\{b_{kj} : j < n\} \cup \{b_{kl}\}$  such that

$$\text{trdeg}(b_{kl'} \mid c, b_{kl}) = \text{trdeg}(b_{kl'} \mid b_{i_1 j_1}, \dots, b_{i_m j_m}, b_{k j'_1}, \dots, b_{k j'_{m'}}).$$

As the row  $b_k$  is indiscernible over  $\{b_{ij} : i \neq k\}$ , it follows that

$$\text{trdeg}(b_{kl'} \mid c, b_{kl}) = \text{trdeg}(b_{kl'} \mid b_{i_1 j_1}, \dots, b_{i_m j_m}, b_{k1}, \dots, b_{k m'}),$$

which is  $\geq \text{trdeg}(b_{kl'} \mid c)$ . Therefore,  $\text{trdeg}(b_{kl'} \mid c, b_{kl}) = \text{trdeg}(b_{kl'} \mid c)$ , and so  $b_{kl'}$  is independent with  $b_{kl}$  over  $c$ . Rename  $b_{i(j+n)}$  as  $b_{i,j}$  for  $i, j \in \omega$  we achieved the desired goal.

Replace  $c$  by a listing of elements in  $\text{acl}_r^\times(c)$ , we obtain (2) without losing (4). By compactness, Erdős-Rado theorem and saturation of  $F^\times$ , we can arrange to have (5) and (6) without losing (2) and (4).

We next arrange to have (1). If the  $k$ -th term and  $k'$ -term are the same for some  $(i, j) \in \kappa \times \lambda$  then that happens for all  $(i, j) \in \kappa \times \lambda$  by (5) and (6). We can arrange that there is no repetition of the terms of  $b_{ij}$  for all  $(i, j) \in \kappa \times \lambda$  by changing  $\varphi$  if necessary. If a term of  $c$  appears as the  $k$ -th term in  $b_{ij}$  for some  $(i, j) \in \kappa \times \lambda$ , then that happens for all  $(i, j) \in \kappa \times \lambda$  by (5) and (6). We can arrange that for every  $(i, j) \in \kappa \times \lambda$ , the last  $n' < n$  terms of  $b_{ij}$  are terms of  $c$  while the first  $n - n'$  terms of  $b_{ij}$  are not terms of  $c$ . Then by replacing  $n$  by  $n - n'$ , replacing  $b_{ij}$  with  $b_{ij} \upharpoonright n$  for each  $(i, j) \in \kappa \times \lambda$ , replacing  $\varphi(x, y)$  by  $\varphi(x, y, y')$  with  $|y| = n$  and  $|y'| = n'$ , permuting  $c$ , we obtain (1). We observe that we did not lose (2), (4), (5), (6).

Finally we modify  $b$  to get (3). Replace  $b_{ij}$  by a listing of elements in  $\text{acl}_r^\times(c, b_{ij}) \setminus \text{acl}_r^\times(c)$  to get (3). Observe that (1), (2), (4) are preserved. We might lose (5) and (6) in the procedure but that can be easily recovered by standard methods.  $\square$

**Lemma 5.2.** *Suppose  $\text{ACFO}^\times$  have  $\text{TP}_2$ . Then we can find  $\varphi^\times(x, y, y')$ ,  $n, n'$ ,  $(F^\times; \mathcal{A}, <, \mathcal{P})$ ,  $b, c$  as in the preceding lemma and a non-constant  $\mu$ -sequence  $a$  with terms in  $(F^\times)^\omega$  with the following properties:*

- (1) For all  $i \in \kappa$  and  $k \in \mu$ ,  $\varphi^\times(a_k \upharpoonright 1, b_{i0} \upharpoonright n, c \upharpoonright n')$ ;
- (2) For all  $k \in \mu$ ,  $a_k$  is an listing of  $\text{acl}_r^\times(c, a_k \upharpoonright 1) \setminus \text{acl}_r^\times(c)$ ;
- (3)  $a$  is indiscernible over  $\{b, c\}$ ;
- (4)  $(b_{i0})_{i \in \kappa}$  is indiscernible over  $\{a, c\}$ ;

*Proof.* Suppose  $\varphi^\times(x, y, y'), n, n', (F^\times; \mathcal{A}, <, \mathcal{P}), b, c$  are as in the preceding lemma. We first show there is an infinite  $\mu$ -sequence  $a$  with terms in  $F^\times$  such that for all  $i \in \kappa$  and  $k \in \mu$ , we have  $\varphi^\times(a_k, b_{i0} \upharpoonright n, c \upharpoonright n')$ . Since  $b$  as a sequence of its rows is indiscernible by (5) of Lemma 5.1, it suffices to show the previous statement with  $\kappa$  replaced by  $\kappa_{>0} = \{i \in \kappa : i > 0\}$ . By (2) of Lemma 5.1  $(b_{ij} \upharpoonright n, c \upharpoonright n')_{(i,j) \in \kappa \times \lambda}$  is a witness for the  $\text{TP}_2$  of  $\varphi^\times(x, y)$ . Hence, for each  $j \in \lambda$  we have  $X_{0j} \cap \bigcap_{i \in \kappa, i > 0} X_{i0}$  is non-empty, where  $X_{ij}$  is the set defined by  $\varphi^\times(x, b_{ij} \upharpoonright n, c \upharpoonright n')$ . On the other hand  $X_{0j} \cap X_{0j'} \neq \emptyset$  for distinct  $j, j' \in \lambda$ . The conclusion follows.

From the preceding paragraph, it is easy to obtain an infinite  $\mu$ -sequence  $a$  satisfying (1) and (2). Modifying  $a, b$ , using compactness, saturatedness of  $F$  and Erdős-Rado theorem in the same fashion as in Lemma 5.1, we get  $a, b$  satisfy (3) and (4).  $\square$

Still under the assumption that  $\text{ACFO}^\times$  has  $\text{TP}_2$ . We will show that  $\text{ACF}^\times$  and  $T_m$  have  $\text{NTP}_2$ . From this, we deduce the existence of special “counter-witnesses” of a special type for the fact that the construction in Lemma 5.1 is not a witness for the  $\text{TP}_2$  of  $\text{ACF}^\times$  or  $T_m$ .

**Lemma 5.3.**  *$\text{ACF}^\times$  and  $T_m$  have  $\text{NTP}_2$ .*

*Proof.* The theory  $\text{ACF}$  is stable and hence has  $\text{NTP}_2$ . As  $\text{NTP}_2$  is preserved under taking reduct, the desired conclusion for  $\text{ACF}^\times$  follows from the fact that every model of  $\text{ACF}^\times$  is interpretable in a model of  $\text{ACF}$ .

As every ordered abelian group has NIP, the theory  $T_a$  has NIP and so has  $\text{NTP}_2$ . We also know that every model of  $T_m$  is interpretable in a model of  $T_a$ . The conclusion for  $\text{NTP}_2$  thus follows.  $\square$

**Lemma 5.4.** *Suppose  $\text{ACFO}^\times$  has  $\text{TP}_2$  and  $\varphi^\times(x, y, y'), n, n', (F^\times; \mathcal{A}, <, \mathcal{P}), a, b, c$  are as in the preceding lemma. Then there are  $d, e \in (F^\times)^\omega$  with the following properties:*

- (1)  $\text{tp}_r(a_0, b_{00} \mid c) = \text{tp}_r(d, b_{01} \mid c) = \text{tp}_r(d, b_{02} \mid c)$ ;
- (2)  $d \upharpoonright 1 \notin \text{acl}_r^\times(c, b_{01}, b_{02})$ ;
- (3)  $\text{tp}_m(a_0, b_{00} \mid c) = \text{tp}_m(e, b_{01} \mid c) = \text{tp}_m(e, b_{02} \mid c)$ ;
- (4)  $\text{acl}_m(c, e) \cap \text{acl}_r^\times(c, b_{01}, b_{02}) = \text{acl}_r^\times(c)$ .

*Proof.* Throughout the proof,  $\varphi^\times(x, y, y'), n, n', (F^\times; \mathcal{A}, <, \mathcal{P}), a, b, c$  are as given. We first note that the terms of the sequence  $(a_k \upharpoonright 1)_{k \in \mu}$  are algebraically independent over  $c$ . Suppose the preceding statement does not hold. By (3) of Lemma 5.2,  $a$  is indiscernible over  $c$ . Therefore, the sequence  $(a_k \upharpoonright 1)_{k \in \mu}$  is constant. By (2) of Lemma 5.2, it is easy to see that  $a$  is a constant sequence, a contradiction to the definition of  $a$ .

We also note that for all distinct  $k, k' \in \mu$ ,  $a_k$  and  $a_{k'}$  has no common terms. Suppose otherwise. Then by (2) of Lemma 5.2 and exchange principle for algebraic independence, we obtain algebraic dependency of  $(a_k \upharpoonright 1)_{k \in \mu}$  which is a contradiction to the first paragraph.

We construct  $f \in (F^\times)^{\mu \times \omega}$  with  $\text{tp}_r(f, b_{01} \mid c) = \text{tp}_r(f, b_{02} \mid c) = \text{tp}_r(a, b_{00} \mid c)$ . Let  $\mathbf{p}_r(w, y, c)$  be  $\text{tp}_r(a, b_{00} \mid c)$  noting that  $w$  is an  $\mu \times \omega$ -tuple of variables and  $y$  is an  $\omega$ -tuple of variables. From (4) of Lemma 5.2,  $(b_{i0})_{i \in \kappa}$  is indiscernible over  $\{c\} \cup \{a\}$ . Hence,  $\mathbf{p}_r(w, y, c)$  is also  $\text{tp}_r(a, b_{i0} \mid c)$ . Therefore  $\bigcup_{i \in \kappa} \mathbf{p}_r(w, b_{i0}, c)$  defines a non-empty subset of  $(F^\times)^{\mu \times \omega}$ , which in particular contains  $a$ . By (5) of Lemma 5.1, the row  $b_i$  is indiscernible over  $\{c\} \cup \{b_{i'} : i' \in \kappa, i' \neq i\}$  for all  $i \in \kappa$ . As a consequence, we also have that

$$\bigcup_{i \in \kappa} \mathbf{p}_r(w, b_{ij(i)}, c)$$

defines non-empty subset of  $(F^\times)^{\mu \times \omega}$  for any  $j : \kappa \rightarrow \lambda$ . As ACF has  $\text{NTP}_2$ , for all formula  $\psi(w, y, c) \in \mathbf{p}_r(w, y, c)$ , there are  $i \in \kappa$  and distinct  $j, j' \in \lambda$ ,  $\psi(z, b_{ij}, c)$  and  $\psi(z, b_{ij'}, c)$  defines a non-empty set. By mutual discernibility of the rows of  $b$ , we can arrange that  $i = 0$  and  $j = 1, j' = 2$ . Hence, by compactness and saturatedness of  $F$ , there is a  $\mu$ -sequence  $f$  with terms in  $(F^\times)^\omega$  which realizes  $\mathbf{p}_r(w, b_{01}, c) \cup \mathbf{p}_r(w, b_{02}, c)$ .

From (3) of Lemma 5.2,  $a$  is indiscernible over  $\{c\} \cup \{b_{00}\}$ . Hence, for all  $k \in \mu$ ,  $\text{tp}_r(a_k, b_{00} \mid c) = \text{tp}_r(a_0, b_{00} \mid c)$ . Therefore, with  $f$  as constructed in the preceding paragraph and an arbitrary  $k \in \mu$ , we have that

$$\text{tp}_r(f_k, b_{01} \mid c) = \text{tp}_r(f_k, b_{02} \mid c) = \text{tp}_r(a_0, b_{00} \mid c).$$

It follows from the first paragraph that the terms of  $(f_k \upharpoonright 1)_{k \in \mu}$  is algebraically independent over  $c$ . We can arrange that  $\mu$  is large compared to  $\omega$ . Hence, there is  $k$  such that  $f_k$  is not in  $\text{acl}_r^\times(c, b_{01}, b_{02})$ . We set  $d = f_k$  and check that condition (1) and (2) are satisfied.

Using the  $\text{NTP}_2$  of  $T_{m,p}$  and a similar method as in the second paragraph, we obtain  $g \in (F^\times)^{\mu \times \omega}$  with  $\text{tp}_m(g, b_{01} \mid c) = \text{tp}_m(g, b_{02} \mid c) = \text{tp}_m(a, b_{00} \mid c)$ . For arbitrary  $k \in \mu$ , we have that

$$\text{tp}_m(g_k, b_{01} \mid c) = \text{tp}_m(g_k, b_{02} \mid c) = \text{tp}_m(a_0, b_{00} \mid c).$$

We again arrange that  $\mu$  is large compared to  $\omega$ . It follows from the second paragraph that for distinct  $k, k' \in \mu$ ,  $g_k$  and  $g_{k'}$  have no common terms. Then by the preceding paragraph, there is  $k \in \mu$  such that  $\text{acl}_r^\times(c, b_{01}, b_{02})$  contains no term of  $g_k$ . Choose  $k$  with such property and set  $e$  to be  $g_k$ . We note that  $\text{tp}_m(g_k \mid c) = \text{tp}_m(a_0 \mid c)$  implies that  $(c, g_k)$  is a listing of all elements in  $\text{acl}_m(c, e)$ . The properties (3) and (4) are then satisfied by construction.  $\square$

Still under the assumption that  $\text{ACFO}^\times$  has  $\text{TP}_2$ , we “glue” the “counter-witnesses” in the previous part to obtain a “counter-witness” for the fact that the construction in Lemma 5.1 is not a witness for the  $\text{TP}_2$  of  $\text{ACFO}^\times$ . This is the contradiction we want.

We need some auxiliary results which is of some independent interest. Let  $(M_i)_{i \in I}$  be a family of subgroups of an abelian multiplicative group  $M$ . Then  $(M_i)_{i \in I}$  is **weakly disjoint** if for all  $i_1, \dots, i_k \in I$  and  $a_1 \in M_{i_1}, \dots, a_k \in M_{i_k}$  with  $a_1 \cdots a_k = 1$ , there are  $a'_2 \in M_{i_1} \cap M_{i_2}, \dots, a'_k \in M_{i_1} \cap M_{i_k}$  such that  $a_1 a'_2 \cdots a'_k = 1$ . If  $(M_i)_{i \in I}$  consists of  $M_1, \dots, M_n$ , we also say  $M_1, \dots, M_n$  are weakly disjoint. We note the reader that any two multiplicative subgroup of a multiplicative group are automatically weakly disjoint.

**Lemma 5.5.** *Let  $b_1, \dots, b_n$  be finite tuple of elements in  $F$  which are relatively algebraically independent over  $c$ . For each  $S \subseteq \{b_1, \dots, b_n\}$ , let  $F_S$  field with underlying set  $\text{acl}_r(\{c\} \cup S)$ . Then the family  $(F_S^\times)_{S \subseteq \{b_1, \dots, b_n\}}$  is weakly disjoint.*

*Proof.* Suppose  $b_1, \dots, b_n, c, F_S$  are as in the statement of the lemma. For arbitrary  $S_1, \dots, S_k \subseteq \{b_1, \dots, b_n\}$  and  $a_1 \in F_{S_1}^\times, \dots, a_k \in F_{S_k}^\times$  with  $a_1 \cdots a_k = 1$ , we need to find

$$a'_2 \in F_{S_1}^\times \cap F_{S_2}^\times = F_{S_1 \cap S_2}^\times, \dots, a'_k \in F_{S_1}^\times \cap F_{S_k}^\times = F_{S_1 \cap S_k}^\times$$

such that  $a_1 a'_2 \cdots a'_k = 1$ . We note that  $F_{S_1}^\times \cap F_{S_i}^\times = F_{S_1 \cap S_i}^\times$  for  $i \in \{2, \dots, k\}$  as algebraic independence satisfies exchange property.

We make a number of preparations. Let  $S_1^c = \{b_1, \dots, b_n\} \setminus S_1$ . By permuting  $b_1, \dots, b_n$  if needed, we can arrange that  $b_1, \dots, b_l \in S_1$  and  $b_{l+1}, \dots, b_n \in S_1^c$  for  $0 \leq l \leq n$ . Let  $F_0$  be field with underlying set  $\text{acl}_r(0)$ . For  $i \in \{1, \dots, k\}$ , let  $P_i \in F_0[x_i, y_1, \dots, y_n]$  be such that  $P_i(x, b_1, \dots, b_n)$  is in  $F_0[S_i]$  and is the minimal polynomial of  $a_i$  over  $F_0(S_i)$ . Let  $\varphi$  be in  $L_{r, F_0}(y_1, \dots, y_n)$  such that for all  $b'_1, \dots, b'_n \in F$ ,  $\varphi(b'_1, \dots, b'_n)$  if and only if for all  $i \in \{1, \dots, k\}$ ,

- (1)  $P_i(x_i, b'_1, \dots, b'_n)$  is not a constant polynomial, and
- (2) there are  $a'_1, \dots, a'_k \in F^\times$  with  $a'_1 \cdots a'_k = 1$  and  $P_i(a'_i, b'_1, \dots, b'_n) = 0$ .

Let  $\mathbf{q}$  be the type of  $(b_1, \dots, b_l)$  over  $F_{S_1^c}$ .

We now produce  $a'_2, \dots, a'_k$  as prescribed in the first paragraph. As ACF is stable,  $\mathbf{q}$  is definable over  $F_{S_1^c}$ . Moreover,  $\mathbf{q}$  is the non-forking extension of the type  $\mathbf{p}$  of  $(b_1, \dots, b_l)$  over  $F_0$ , so  $\mathbf{q}$  is definable over  $F_0$ . Hence, we can find  $\psi \in L_{r, F_0}(y_{l+1}, \dots, y_n)$  such that for all  $b'_{l+1}, \dots, b'_n \in F_{S_1^c}$

$$\varphi(x_1, \dots, x_l, b'_{l+1}, \dots, b'_n) \in \mathbf{q} \Leftrightarrow \psi(b'_{l+1}, \dots, b'_n).$$

Therefore, we can find  $b'_{l+1}, \dots, b'_n \in F_0$  such that  $\varphi(b_1, \dots, b_l, b'_{l+1}, \dots, b'_n)$  by model completeness of ACF. Let  $a'_1, \dots, a'_k$  be as in (2) of the definition of  $\varphi$ . We have  $a'_1$  is a conjugate of  $a_1$  over  $F_0(S_1)$  and  $a'_i$  is in  $F_{S_1 \cap S_i}^\times$  for all  $i \in \{2, \dots, k\}$ . Let  $\sigma$  be an automorphism of  $F_{S_1}$  over  $F_0(S_1)$  sending  $a'_1$  to  $a_1$ . Then  $a_1 \sigma(a_2)' \cdots \sigma(a_k)' = 1$ . We check that  $\sigma(a_2)', \dots, \sigma(a_k)'$  are as desired.  $\square$

We are particularly interested in the case when there are three groups involved.

**Lemma 5.6.** *If  $M_1, M_2, M_3$  are subgroups of a multiplicative abelian group  $M$ , then the following are equivalent:*

- (1)  $M_1, M_2, M_3$  are weakly disjoint;
- (2) if  $a_1 \in M_1, a_2 \in M_2, a_3 \in M_3$  are such that  $a_1 a_2 a_3 = 1$ , then  $a_1 = b_2 b_3^{-1}, a_2 = b_3 b_1^{-1}, a_3 = b_1 b_2^{-1}$  with  $b_1 \in M_2 \cap M_3, b_2 \in M_3 \cap M_1, b_3 \in M_1 \cap M_2$ ;
- (3)  $M_1 \cap \langle M_2, M_3 \rangle = \langle M_1 \cap M_2, M_1 \cap M_3 \rangle$ ;

*Proof.* It is immediate that (1) implies (3) and (2) implies (1) is clear. We show that (3) implies (2). Suppose (3) and  $a_1 a_2 a_3 = 1$  with  $a_1 \in M_1, a_2 \in M_2, a_3 \in M_3$ . Then  $a_1 = a_2^{-1} a_3^{-1}$  is in  $M_1 \cap \langle M_2, M_3 \rangle$ . Then there are  $a'_2 \in M_1 \cap M_2$  and  $a'_3 \in M_1 \cap M_3$  such that  $a_1 = a'_2 a'_3$ . Set  $b_1 = (a_2)^{-1} (a'_2)^{-1} = a_3 a'_3, b_2 = a'_3, b_3 = (a'_2)^{-1}$ . It is easy to check all the desired requirements.  $\square$

**Lemma 5.7.** *Suppose  $\text{ACFO}^\times$  has  $\text{TP}_2$  and  $\varphi^\times(x, y, y')$ ,  $n, n', (F^\times; \mathcal{A}, <, \mathcal{P})$ ,  $a, b, c, d, e$  are as in Lemma 5.4. Then we can find  $f_1, f_2 : \text{acl}_r^\times(a_0, b_{00}, c) \rightarrow F^\times$  and  $g_1 : \text{acl}_r^\times(d, b_{01}, c) \rightarrow \text{acl}_r^\times(a, b_{00}, c)$ ,  $g_2 : \text{acl}_r^\times(d, b_{02}, c) \rightarrow \text{acl}_r^\times(a, b_{00}, c)$  with the following properties:*

- (1) *For  $i \in \{1, 2\}$ ,  $f_i$  is an  $L_m$ -embedding with  $(a_0, b_{00}, c) \mapsto (e, b_{0i}, c)$ ;*
- (2) *with  $N_0 = \text{acl}_r^\times(b_{01}, b_{02}, c)$ ,  $N_i = f_i(\text{acl}_r^\times(a, b_{00}, c))$  for  $i \in \{1, 2\}$ , we have  $N_0 \cap N_i = \text{acl}_r^\times(b_{0i}, c)$  for  $i \in \{1, 2\}$ ,  $N_1 \cap N_2 = \text{acl}_r^\times(e, c)$  and  $N_0, N_1, N_2$  are weakly disjoint.*
- (3) *for  $i \in \{1, 2\}$ ,  $g_i$  is an  $L_r^\times$ -isomorphism with  $(d, b_{0i}, c) \mapsto (a_0, b_{00}, c)$ .*

*Proof.* Suppose  $\varphi^\times(x, y, y')$ ,  $n, n', (F^\times; \mathcal{A}, <, \mathcal{P})$ ,  $a, b, c, d, e, N_0, N_1, N_2$  are as stated. We construct  $f_1$  and  $f_2$ . Choose a multiplicative basis  $h$  of  $\text{acl}_r^\times(a_0, b_{00}, c)$  over the multiplicative subgroup  $\text{acl}_m(a_0, b_{00}, c)$ . We note that  $\text{acl}_r^\times(a_0, b_{00}, c) = \text{acl}_m(a_0, b_{00}, h, c)$ . We will find a tuple  $h'$  with terms in  $F^\times$  with the following properties:

- (1)  $\text{tp}_m(a_0, b_{00}, h, c) = \text{tp}_m(e, b_{01}, h', c)$ ;
- (2)  $h'$  is multiplicatively independent over  $\langle \text{acl}_m(e, b_{01}, c), N_0 \rangle$ ,

where the latter is the subgroup of  $F^\times$  generated by  $\text{acl}_m(e, b_{01}, c)$  and  $N_0$ . From (3) of Lemma 5.4, we have  $\text{tp}_m(a_0, b_{00}, c) = \text{tp}_m(e, b_{01}, c)$ . By saturation of  $F$ , there is  $h'$  in  $F^\times$  satisfy (1). In particular,  $h'$  is multiplicatively independent over  $\text{acl}_m(e, b_{01}, c)$ . By Lemma 4.13, we can arrange that  $h'$  also satisfy (2). We then set  $f_1 : \text{acl}_m(a_0, b_{00}, c, h) \rightarrow F^\times$  to be an  $L_m$ -embedding such that

$$(a_0, b_{00}, c, h) \mapsto (e, b_{01}, c, h') \text{ and } N_1 = f_1(\text{acl}_r^\times(a_0, b_{00}, c)) = \text{acl}_m(e, b_{01}, c, h').$$

Likewise, we can find tuple  $h''$  with terms in  $F^\times$  with the following properties:

- (1)  $\text{tp}_m(a_0, b_{00}, h, c) = \text{tp}_m(e, b_{02}, h'', c)$ ;
- (2)  $h''$  is multiplicatively independent over  $\langle \text{acl}_m(e, b_{02}, c), N_0, N_1 \rangle$ .

where the later is the obvious subgroup of  $F^\times$ . Similarly, we obtain an  $L_m$ -embedding  $f_2 : \text{acl}_m(a_0, b_{00}, c, h) \rightarrow \text{acl}_m(e, b_{02}, c, h'')$  such that

$$(a_0, b_{00}, c, h) \mapsto (e, b_{02}, c, h'') \text{ and } M_2 = f_2(\text{acl}_r^\times(a_0, b_{00}, c)) = \text{acl}_m(e, b_{02}, c, h'').$$

From the above construction, it easily follows that (1) is satisfied. We check that  $N_0 \cap N_i = \text{acl}_r^\times(b_{0i}, c)$  for  $i \in \{1, 2\}$  and  $N_1 \cap N_2 = \text{acl}_r^\times(e, c)$ . From above,

$$N_0 \cap N_1 = \text{acl}_r^\times(b_{01}, b_{02}, c) \cap \text{acl}_m(e, b_{01}, c, h')$$

As  $h'$  is multiplicative independent over  $\langle \text{acl}_m(e, b_{01}, c), M_0 \rangle$ , the above is equal to  $\text{acl}_r^\times(b_{01}, b_{02}, c) \cap \text{acl}_m(e, b_{01}, c)$  which is  $\text{acl}_m(b_{01}, c)$  by (4) of lemma 5.4. A similar verification applies to  $N_0 \cap N_2$ .

We next check that  $N_1 \cap N_2 = \text{acl}_r^\times(e, c)$ . From the definition, we have:

$$N_1 \cap N_2 = \text{acl}_m(e, b_{01}, c, h') \cap \text{acl}_m(e, b_{02}, c, h'').$$

As  $h''$  is multiplicative independent over  $\langle \text{acl}_m(e, b_{02}, c), N_1 \rangle$ , the above is equal to  $\text{acl}_m(e, b_{01}, c, h') \cap \text{acl}_m(e, b_{02}, c)$ . This is equal to  $\text{acl}_m(e, b_{01}, c) \cap \text{acl}_m(e, b_{02}, c)$  as  $h'$  is multiplicative independent over  $\langle \text{acl}_m(e, b_{01}, c), N_0 \rangle$ . For  $i \in \{1, 2\}$ , we note that  $\text{acl}_m(e, b_{0i}, c) = \langle \text{acl}_m(e, c), \text{acl}_m(b_{0i}, c) \rangle$  because the latter is divisible and contains all torsion points of  $F^\times$ . Hence, by (4) of Lemma 5.4 and (4) of Lemma 5.1,  $\text{acl}_m(e, b_{01}, c) \cap \text{acl}_m(e, b_{02}, c) = \text{acl}_m(e, c)$ .

We check that  $N_2 \cap \langle N_0, N_1 \rangle = \langle N_2 \cap N_0, N_2 \cap N_1 \rangle$  which shows that  $N_0, N_1, N_2$  are weakly disjoint by Lemma 5.6. We have

$$N_2 \cap \langle N_0, N_1 \rangle = \text{acl}_m(e, b_{02}, c, h'') \cap \langle \text{acl}_m(e, b_{01}, c, h'), \text{acl}_r^\times(b_{01}, b_{02}, c) \rangle.$$

As  $h''$  is multiplicatively independent over  $\langle \text{acl}_m(e, b_{02}, c), N_0, N_1 \rangle$  the expression on the right hand side is equal to  $\text{acl}_m(e, b_{02}, c) \cap \langle \text{acl}_m(e, b_{01}, c, h'), \text{acl}_r^\times(b_{01}, b_{02}, c) \rangle$ . We have proven above that  $\text{acl}_m(e, b_{02}, c) = \langle \text{acl}_m(e, c), \text{acl}_m(b_{02}, c) \rangle$  which is clearly a subset of  $\langle \text{acl}_m(e, b_{01}, c, h'), \text{acl}_r^\times(b_{01}, b_{02}, c) \rangle$ . As a consequence,

$$N_2 \cap \langle N_0, N_1 \rangle = \langle \text{acl}_m(e, c), \text{acl}_m(b_{02}, c) \rangle,$$

which is equal to  $\langle M_2 \cap M_1, M_2 \cap M_0 \rangle$  by the preceding paragraph.

The existence of  $g_1, g_2$  with the desired property follows immediately from (1) of Lemma 5.4.  $\square$

**Lemma 5.8.** *Suppose  $\text{ACFO}^\times$  have  $\text{TP}_2$  and  $\varphi^\times(x, y, y')$ ,  $n, n', (F^\times; \mathcal{A}, <, \mathcal{P})$ ,  $a, b, c, d, e, f_1, f_2, g_1, g_2, N_0, N_1, N_2$  are as in the preceding lemma. Set*

$$M_0 = \text{acl}_r^\times(b_{01}, b_{02}, c), \quad M_i = \text{acl}_r^\times(d, b_{0i}, c) \text{ for } i \in \{1, 2\} \text{ and } M = \langle M_0, M_1, M_2 \rangle.$$

*Then there is a unique  $L_g$ -embedding  $h : M \rightarrow F^\times$  such that for arbitrary choice of  $\alpha_0 \in M_0, \alpha_1 \in M_1$  and  $\alpha_2 \in M_2$ , we have*

$$h(\alpha_0 \alpha_1 \alpha_2) = \alpha_0(f_1 \circ g_1(\alpha_1))(f_2 \circ g_2(\alpha_2)).$$

*Define  $\mathcal{A}$  on  $M$  as the restriction of  $\mathcal{A}$  on  $F^\times$ ,  $<'$  and  $\mathcal{P}'$  on  $M$  as the pullback of  $<$  and  $\mathcal{P}$  on  $F^\times$  by  $h$ . Then we have that  $(M; \mathcal{A}) \models \text{ACF}_p^\times(\forall)$  and  $(M; <', \mathcal{P}') \models T_{m,p}(\forall)$  where  $p = \text{char}(F)$ . Moreover, the following are  $L_c^\times$ -isomorphism:*

$$\iota_0 : (M_0; \mathcal{A}, <', \mathcal{P}') \rightarrow (M_0; \mathcal{A}, <, \mathcal{P}), \quad \alpha_0 \mapsto \alpha_0$$

$$\iota_i : (M_i; \mathcal{A}, <', \mathcal{P}') \rightarrow (\text{acl}_r^\times(a_0, b_{0i}, c); \mathcal{A}, <, \mathcal{P}), \quad \alpha_i \mapsto g_i(\alpha_i) \text{ with } i \in \{1, 2\}.$$

*Proof.* Suppose the notations are as in the lemma. We first make a number of observations. As algebraic independence satisfies exchange property, we have

$$M_1 \cap M_2 = \text{acl}_r^\times(d, c) \text{ and } M_0 \cap M_i = \text{acl}_r^\times(b_{0i}, c) \text{ for } i \in \{1, 2\}$$

By (2) of Lemma 5.2 and (1) of Lemma 5.4, every elements of  $\text{acl}_r^\times(d, c)$  appears as terms of either  $c$  or  $d$ . By (1) and (3) of Lemma 5, under both  $f_1 \circ g_1$  and  $f_2 \circ g_2$ ,  $(d, c) \mapsto (e, c)$ . Hence,  $f_1 \circ g_1$  and  $f_2 \circ g_2$  agrees on  $M_1 \cap M_2$ . Likewise, using (3) of Lemma 5.1 and arguing similarly, we get  $f_i \circ g_i$  agree with the identity map on  $M_0 \cap M_i$  for  $i \in \{1, 2\}$ .

To get a group homomorphism  $h$  satisfying the desired equation, we need to check for  $\alpha_0, \alpha'_0 \in M_0, \alpha_1, \alpha'_1 \in M_1$  and  $\alpha_2, \alpha'_2 \in M_2$  with  $\alpha_0 \alpha_1 \alpha_2 = \alpha'_0 \alpha'_1 \alpha'_2$  that

$$\alpha_0(f_1 \circ g_1(\alpha_1))(f_2 \circ g_2(\alpha_2)) = \alpha'_0(f_1 \circ g_1(\alpha'_1))(f_2 \circ g_2(\alpha'_2)).$$

By (3) of Lemma 5.1,  $M_0 = \text{acl}_r^\times(b_{01} \upharpoonright n, b_{02} \upharpoonright n, c)$ . From (2) of Lemma 5.2 and (1) of Lemma 5.4,  $M_i = \text{acl}_r^\times(d \upharpoonright 1, b_{0i} \upharpoonright n, c)$  for  $i \in \{1, 2\}$ . By (4) of Lemma 5.1 and (2) of Lemma 5.4,  $d \upharpoonright 1, b_{01} \upharpoonright n, b_{02} \upharpoonright n$  are mutually algebraic independent so  $M_0, M_1, M_2$  are weakly disjoint by Lemma 5.5. So  $\alpha_0 \alpha_1 \alpha_2 = \alpha'_0 \alpha'_1 \alpha'_2$  implies there are  $\beta_0 \in M_1 \cap M_2, \beta_1 \in M_0 \cap M_2$  and  $\beta_2 \in M_0 \cap M_1$  such that

$$\alpha'_1 = \alpha_1 \beta_2 \beta_3^{-1}, \alpha'_2 = \alpha_2 \beta_3 \beta_1^{-1} \text{ and } \alpha'_3 = \alpha_3 \beta_1 \beta_2^{-1}.$$

The desired conclusion follows from the observations in the preceding paragraph and a straightforward calculation.

We next show that the group homomorphism  $h$  in the preceding paragraph is an  $L_g$ -embedding. Suppose  $h(\alpha_0\alpha_1\alpha_2) = 1$ . We note that

$$\alpha_0 \in N_0, f_1 \circ g_1(\alpha_1) \in N_1 \text{ and } f_2 \circ g_2(\alpha_2) \in N_2.$$

By the preceding lemma,  $N_0, N_1, N_2$  are weakly disjoint. By Lemma 5.6

$$\alpha_0 = \beta_1\beta_2^{-1}, \alpha_1 = (f_1 \circ g_1)^{-1}(\beta_2\beta_1^{-1}) \text{ and } \alpha_2 = (f_2 \circ g_2)^{-1}(\beta_0\beta_1^{-1}).$$

Lemma 5(2) and an argument similar to that given in the first paragraph shows that  $(f_1 \circ g_1)^{-1}$  agrees with  $(f_2 \circ g_2)^{-1}$  on  $N_0$  and  $(f_i \circ g_i)^{-1}$  agrees with identity map on  $N_i$  for  $i \in \{1, 2\}$ . The conclusion then follows from a straightforward calculation.

The remaining conclusions are obvious from the construction.  $\square$

*Proof of Theorem 1.9 .* It suffices to show that  $\text{ACFO}^\times$  has  $\text{NTP}_2$ . Suppose otherwise. Then there are  $\varphi^\times(x, y, y')$ ,  $n, n', (F^\times; \mathcal{A}, <, \mathcal{P})$ ,  $a, b, c, d, e, f_1, f_2, g_1, g_2, M, M_0, M_1, M_2, N_0, N_1, N_2, \iota_0, \iota_1, \iota_2$  as in the preceding lemma. In particular,  $(M; \mathcal{A}, <', \mathcal{P}')$  has  $(M; \mathcal{A}) \models \text{ACF}_p^\times$  and  $(M; <', \mathcal{P}') \models T_{m,p}(\forall)$  where  $p = \text{char}(F)$ . By Proposition 4.10  $(M; \mathcal{A}, <', \mathcal{P}')$  has an  $L_c^\times$ -extension

$$((F')^\times; \mathcal{A}, <', \mathcal{P}') \text{ which is the } L_c^\times\text{-reduct of } (F'; <', \mathcal{P}') \models \text{ACFO}_p.$$

We can also arrange that  $(F'; <', \mathcal{P}')$  is an  $L_c$ -extension of  $(F_0; <', \mathcal{P}') \models \text{ACFO}^\times$  where  $F_0$  is the subfield of  $F$  with underlying set  $\text{acl}_F(b_{01}, b_{02}, c)$ . We note that  $(F_0; <', \mathcal{P}')$  is equal to  $(F_0; <, \mathcal{P})$  as  $\iota_0$  is an  $L_c^\times$ -isomorphism. As  $\text{ACFO}$  is the model completion of  $\text{ACFO}^\times$ , we can find an  $L_c$ -embedding  $j$  of  $(F'; <', \mathcal{P}')$  into  $(F; <, \mathcal{P})$  over  $(F_0; <, \mathcal{P})$ . Then,  $(a_0, b_{00}, c) \mapsto (j(d), b_{01}, c)$  and

$$(\text{acl}_F^\times(a_0, b_{00}, c); \mathcal{A}, <, \mathcal{P}) \cong_{L_c^\times} (\text{acl}_F^\times(j(d), b_{01}, c); \mathcal{A}, <, \mathcal{P})$$

under the map  $j \circ \iota_1^{-1}$ . Therefore,  $\text{tp}(a_0, b_{00}, c) = \text{tp}(j(d), b_{01}, c)$  by Theorem 1.3. In particular, we have that  $\varphi(j(d) \upharpoonright 1, b_{01} \upharpoonright n, c \upharpoonright n')$ . A similar argument then gives us that  $\varphi(j(d) \upharpoonright 1, b_{02} \upharpoonright n, c \upharpoonright n')$ . This contradicts the assumption (1) of Lemma 5.1 that  $(b_{ij} \upharpoonright n, c \upharpoonright n')_{(i,j) \in \kappa \times \lambda}$  is a witness for the  $\text{TP}_2$  of  $\text{ACFO}^\times$ .  $\square$

*Proof of Proposition 1.10.* Suppose  $(F; <, \mathcal{P}) \models \text{ACFO}$ . For  $b, b' \in F^\times$  such that  $b < b'$ , we let  $[b, b'] = \{a : b \leq a \leq b'\}$ . Fix  $b$  and  $b'$  in  $F^\times$ . Define  $\mathcal{R} \subseteq (F^\times)^2$  by:

$$(a, a') \in \mathcal{R} \text{ if } a \neq a' \text{ and } a + a' \in [b, b'].$$

We will show that  $(F^\times; \mathcal{R})$  is a random graph. For  $m' \leq m$  and distinct elements  $d_1, \dots, d_{m'}, d_{m'+1}, \dots, d_m$  of  $F^\times$ , we need to construct  $e \in F^\times$  such that  $(d_i, e) \in \mathcal{R}$  for  $i \in \{1, \dots, m'\}$  and  $(d_j, e) \notin \mathcal{R}$  for  $j \in \{m' + 1, \dots, m\}$ . Consider the variety

$$V = \{a \in (F^\times)^m : a_i - a_j = d_i - d_j\}.$$

Suppose  $x_1^{k_1} \dots x_m^{k_m}$  is equal to  $c \in F^\times$  on  $V$ . Then we have

$$a_1^{k_1} (a_1 + d_2 - d_1)^{k_2} \dots (a_1 + d_m - d_1)^{k_m} = c \text{ for all } a_1.$$

Hence,  $k_1 = \dots = k_m = 0$ . Therefore,  $V$  is multiplicatively large. Choose  $b_1, b'_1, b_2, b'_2$  such that  $b_1 < b'_1$ ,  $b_2 < b'_2$ ,  $[b_1, b'_1] \subseteq [b, b']$  and  $[b_2, b'_2] \cap [b, b'] = \emptyset$ . As  $V$  is multiplicatively large, we can find

$$a \in V \cap [b_1, b'_1]^{m'} \times [b_2, b'_2]^{m-m'},$$

Let  $e = a_1 - d_1 = \dots = a_m - d_m$ . Then  $e$  satisfies the desired properties by construction. The conclusion follows.  $\square$

## 6. FURTHER QUESTIONS

The results obtained in this paper still leave several open questions about models of ACFO. We expect that ACFO is inp-minimal and have made some progress toward proving this. From Proposition 1.10, any model of ACFO interprets a random graph. Does every model of ACFO interpret a  $(n+1)$ -random  $(n+1)$ -hyper graph for arbitrary  $n > 0$ ? We would also like to obtain more information about definable equivalent relations, definable groups in models of ACFO.

The tameness of models of ACFO suggests related structures might also be tame. Let  $(\mathbb{F}, \mathbb{C}; \chi, \mathbb{R})$  be the two-sorted structures with  $\mathbb{R}$  viewed as a unary relation on  $\mathbb{C}$ . We expect that this structure is tame with the induced structure on  $\mathbb{F}$  bi-interpretable with  $(\mathbb{F}; <)$ . We only consider in this paper structure induced on  $\mathbb{F}$  by an injective multiplicative character  $\chi : \mathbb{F}^\times \rightarrow \mathbb{C}^\times$ . It might also be fruitful to remove the injective assumption, to consider instead additive characters, mixed characters, multiple characters and character into the multiplicative group  $\mathbb{C}_l^\times$  where  $\mathbb{C}_l$  is the valued field of  $l$ -adic complex numbers with  $l$  a prime different from  $\text{char}(\mathbb{F})$ . We are still looking for a structure to apply results in [Kow07]. One candidate is  $\prod_{p \in \mathbb{P}} (\mathbb{F}_p; <_p) / \mathcal{U}$  where  $\mathbb{P}$  is the set of all primes,  $\mathcal{U}$  is an ultra-filter on  $\mathbb{P}$  and  $(\mathbb{F}_p; <_p)$  is the  $L_{\mathbb{C}}$ -structure obtained in similar fashion as  $(\mathbb{F}; <)$  with  $\mathbb{F}$  replaced by  $\mathbb{F}_p$  and  $\chi$  replaced by a group embedding  $\chi_p : \mathbb{F}_p^\times \rightarrow \mathbb{C}^\times$ . However, there are evidences that this structure defines arithmetic.

We end with two vague questions. The tameness of ACFO is a consequence of equidistribution, a very common phenomenon in mathematics. Are there more examples of this type? Are there applications of ACFO in number theory?

## 7. APPENDIX: A MORE ELEMENTARY PROOF OF LEMMA 2.2

We keep the notations in the first paragraphs of section 2 and 3, the paragraph before Lemma 3.10 and moreover assume in this appendix that  $k \geq 1$ ,  $p = \text{char}(\mathbb{F})$ ,  $q = p^l$  for  $l \geq 1$  and  $\mathbb{F}_q$  is the subfield of  $\mathbb{F}$  with  $q$  elements. If  $P$  is a system of polynomials in  $\mathbb{F}[x]$ , let  $Z(P)$  be the zero set of  $P$  in  $\mathbb{F}^m$ . Let  $\mathbb{P}^m$  be the  $m$ -dimension projective space over  $\mathbb{F}$ . A *quasi-projective variety* over  $\mathbb{F}$  is an open subset of an irreducible closed subset of some  $\mathbb{P}^m$ , the latter equipped with its Zariski topology. Let  $V$  ranges over the quasi-affine or quasi-projective varieties over  $\mathbb{F}$  and  $F$  ranges over the subfields of  $\mathbb{F}$ . The set  $V(F)$  of  $F$ -rational points of  $V$  consists of  $a \in V$  with coordinates in  $F$  when  $V$  is quasi-affine and consists of  $a \in V$  which has homogeneous coordinates in  $F$  when  $V$  is quasi-projective; note that  $V$  is not required to be definable over  $F$  in the field sense. If  $V \subseteq \mathbb{P}^m$  is quasi-projective, the  $i$ -th *quasi-affine piece* of  $V$  is the quasi-affine variety  $V \cap U_i$  where  $U_i$  identified with  $\mathbb{F}^m$  is the set of  $a \in \mathbb{P}^m$  with non-zero  $i$ -th homogeneous coordinate.

We say  $V$  is *F-definable* if  $V$  is quasi-affine and definable over  $F$  in  $L_r$ , or if  $V$  is quasi-projective and all affine pieces of  $V$  are  $F$ -definable. In our case, this definition essentially agrees with the field theoretic definition as every algebraic extension of  $F$  is separable.

**Lemma 7.1.** *Let  $G = \text{Gal}(\mathbb{F} \mid F)$ . Then  $G$  acts naturally on  $\mathbb{F}^m$ . For quasi-affine  $V \subseteq \mathbb{F}^m$ , the following are equivalent:*

- (1)  $V$  is  $F$ -definable;
- (2)  $V$  is  $G$  invariant;
- (3) There are systems  $P, Q$  of polynomials in  $F[x]$  such that  $V = Z(P) \setminus Z(Q)$ .



*Proof.* It is immediate that (1) implies (2) and (3) implies (1); we show that (2) implies (3). Suppose  $G$  and  $V$  are as stated and  $V$  is  $G$  invariant. Then  $V = W \setminus T$  where  $W$  and  $T$  are Zariski-closed in  $\mathbb{F}^m$ . We can arrange that  $W$  and  $T$  are also  $G$ -invariant. Let  $F' \subseteq \mathbb{F}$  be a finite Galois extension of  $F$  such that  $W$  is defined by polynomials  $P'_1, \dots, P'_k \in F'[x]$ . Set  $G' = \text{Gal}(\mathbb{F} \mid F')$ , so  $G/G' = \text{Gal}(F' \mid F)$ . Then  $W$  is also defined by the system  $P$  consisting of  $P_1, \dots, P_k \in F[x]$  where

$$P_i = \prod_{\sigma \in G'/G'} \sigma(P'_i) \text{ for } i \in \{1, \dots, k\}.$$

Argue similarly for  $T$ , we get  $Q$ . The desired conclusion follows.  $\square$

Let  $\mathbb{F}(V)$  be the field of  $\mathbb{F}$ -rational functions on  $V$  as usual. Suppose  $V$  is moreover  $F$ -definable. We say  $f \in \mathbb{F}(V)$  is *F-definable* if either  $V$  is quasi-affine and  $f$  is definable over  $F$  in  $L_r$  or if  $V$  is quasi-projective and the restriction of  $f$  to all affine pieces of  $V$  is  $F$ -definable. If  $V$  is quasi-affine, then let  $F(V)$  be the field consisting of the elements  $f$  of  $\mathbb{F}(V)$  such that there are  $P, Q \in F[x]$  with  $Q$  nonzero on  $V$  and  $f = PQ^{-1}$  in  $\mathbb{F}(V)$ . If  $V$  is quasi-projective, let  $F(V)$  be  $F(W)$  where  $W$  is any quasi-affine piece of  $V$ . Again, in this case the model theoretic definition and the field theoretic definition coincides:

**Lemma 7.2.** *Let  $G = \text{Gal}(\mathbb{F} \mid F)$ . For quasi-affine  $V \subseteq \mathbb{F}^m$  definable over  $F$  and  $f \in \mathbb{F}(V)$ , the following are equivalent:*

- (1)  *$f$  is definable over  $F$ ;*
- (2)  *$f$  is  $G$ -invariant under the natural action of  $G$  on  $\mathbb{F}(V)$ ;*
- (3)  *$f$  is in  $F(V)$ .*

*Proof.* It is immediate that (1) implies (2) and (3) implies (1); we show that (2) implies (3). We make a number of preparations. Suppose  $G, V, f$  are as stated and  $f$  is  $G$ -invariant. We can find a finite extension  $F'$  of  $F$  such that  $f$  is in  $F'(V)$ , or in other words,  $f = PQ^{-1}$  in  $\mathbb{F}(V)$  where  $P, Q$  are in  $F'[x_1, \dots, x_m]$  and  $Q$  is nonzero on  $V$ . We note that  $F'$  is automatically a Galois extension of  $F$  as  $G$  is pro-cyclic. Again, set  $G' = \text{Gal}(\mathbb{F} \mid F')$ , so  $G/G' = \text{Gal}(F' \mid F)$

We first consider the case when  $[G : G'] = [F' : F] = n$  with  $p \nmid n$ . Note that

$$f = \frac{1}{n} \sum_{\sigma \in G/G'} \frac{\sigma(P)}{\sigma(Q)}$$

is in  $\mathbb{F}(V)$ . It easily follows that  $f$  is in  $F(V)$ .

We next consider the case when  $[F' : F] = p$ . Then

$$f^p = \prod_{\sigma \in G/G'} \frac{\sigma(P)}{\sigma(Q)}$$

is in  $F(V)$ . On the other hand, as  $V$  is irreducible in  $\mathbb{F}$ ,  $F(V)$  is linearly disjoint with  $F'$  over  $F$ . Therefore,  $[F'(V) : F(V)] = [F' : F]$ . As  $[F' : F]$  is separable,  $[F'(V) : F(V)]$  is also separable. Thus,  $f^p$  is in  $F(V)$  implies  $f$  is in  $F'(V)$ .

For the general case where there is no restriction on  $[F' : F]$ , the conclusion follows the fact that there is a chain of fields

$$F = F'_0 \subseteq \dots \subseteq F'_k = F'$$

such that  $[F'_{i+1} : F'_i]$  is equal to  $p$  or coprime to  $p$  for  $i \in \{0, \dots, k-1\}$ .  $\square$

Let  $\dim(V)$  be the dimension of  $V$  in the sense of algebraic geometry. A *constructible*  $X \subseteq \mathbb{F}^m$  has the form  $V_1 \cup \dots \cup V_k$  where  $V_i \subseteq \mathbb{F}^m$  is a quasi-affine varieties over  $\mathbb{F}$  for all  $i \in \{1, \dots, k\}$ . The dimension  $\dim(X)$  of such  $X$  is defined as  $\max_{i=1}^k \dim(V_i)$ . Constructible subsets of  $\mathbb{P}^m$  and their dimensions are defined similarly replacing quasi-affine varieties with quasi-projective varieties. A constructible  $C \subseteq V$  is a *curve* on  $V$  if  $\dim(C) = 1$ . A curve on  $V$  is *irreducible* if it is moreover a quasi-affine or quasi-projective variety.

Suppose  $C \subseteq \mathbb{F}^m$  is an  $F$ -definable irreducible curve. Then  $\text{trdeg}(F(C) | F) = 1$ . Let  $\tilde{C}(F)$  be the set of all discrete valuations  $v : F(C) \rightarrow \mathbb{Z}$  which has  $v(F) = \{0\}$ . The set  $\mathcal{O}_v = \{f \in F(C) : v(f) \geq 0\}$  is then a subring of  $F(C)$  with maximal ideal  $\{\mathfrak{m}_v = f \in F(C) : v(f) > 0\}$ . The residue field  $F_v = \mathcal{O}_v / \mathfrak{m}_v$  is a finite extension of  $F$ . Set  $\deg(v) = [F_v : F]$ . Given  $f \in F(C)$ , let  $\tilde{Z}_{C,f}(F)$  be the set of  $v \in \tilde{C}(F)$  such that  $v(f) > 0$  and let  $\tilde{P}_{C,f}(F)$  be the set of  $v \in \tilde{C}(F)$  such that  $v(f) < 0$ . For justification of the claims in this paragraph, see [Sti09, Chapter 1].

The main number theoretic ingredient for proving Lemma 2.2 is the following Weil style bound which is a weakening of [Per91, Proposition 4.5]:

**Lemma 7.3.** *Suppose  $C$  is a smooth projective irreducible curve of geometric genus  $g$  definable over  $\mathbb{F}_q$  and  $f \in \mathbb{F}_q(C)$  is not a constant. Then*

$$\left| \sum_{a \in \text{dom} f(\mathbb{F}_q)} \chi(f(a)) \right| \leq \left( 2g - 2 + \sum_{v \in \tilde{Z}_{C,f}(\mathbb{F}_q) \cup \tilde{P}_{C,f}(\mathbb{F}_q)} \deg(v) \right) \sqrt{q}.$$

We note that Lemma 2.2 calls for an upper bound on a character sum over a variety. In view of the preceding lemma, a natural strategy is to obtain a “fibration” of the variety into a family of curves and get an upper bound for the sum over the “fibration” of the right-hand-side expression for each curve. There are two difficulties to carry out this idea: (1) The curves in the “fibration” might not be irreducible or smooth; (2) the right-hand-side expression is not clearly bounded across the “fibration”.

The following lemma is used frequently to show definability of various properties in definable families of sets:

**Lemma 7.4.** *Suppose  $(X_s)_{s \in S}$  is an  $L_r$ -definable family of subsets of  $\mathbb{F}^m$ . There is definable  $S_d \subseteq S$  for  $d \in \mathbb{N}$  and definable  $S_v \subseteq S$  such that for all elementary extension  $\mathbb{F}'$  of  $\mathbb{F}$  and with  $(X'_{s'})_{s' \in S'} = (X_s)_{s \in S}(\mathbb{F}')$ , we have the following:*

- (1)  $S_d(\mathbb{F}') = \{s' \in S' : \dim(X'_{s'}) = d\}$ .
- (2)  $S_v(\mathbb{F}') = \{s' \in S' : X'_{s'} \text{ is a quasi-affine variety over } \mathbb{F}'\}$ .

*Proof.* Suppose  $(X_s)_{s \in S}$ , is as above. As dimension coincides with Morley rank in ACF which is strongly minimal,  $S_d = \{s \in S : \dim(X_s) = d\}$  is definable. We note that if  $X \subseteq \mathbb{F}^m$  has  $\dim(X) = d$ , then there is a definable finite-to-finite relation from  $X$  to  $\mathbb{F}^d$  and so  $\dim(X(\mathbb{F}')) = d$ . The proof that  $S_d$  satisfies (1) follows the same strategy used in the first paragraph of Lemma 3.11.

By Lemma 3.10,  $S_v = \{s \in S : X_s \text{ is a quasi-affine variety over } \mathbb{F}\}$  is definable. Moreover,  $S_v$  satisfies (2) by the the first paragraph of Lemma 3.11.  $\square$

**Corollary 7.5.** *Let  $\mathbb{F}'$  be an elementary extension of  $\mathbb{F}$ . Then  $(C_s)_{s \in S}$  is a definable family of curves on quasi-affine  $V$  if and only if  $(C_s)_{s \in S}(\mathbb{F}')$  is a family of curves on  $V(\mathbb{F}')$ . Moreover,  $(C_s)_{s \in S}$  is a family of irreducible curves on  $V$  if and only if  $(C_s)_{s \in S}(\mathbb{F}')$  is a family of irreducible curves on  $V(\mathbb{F}')$ .*

In the context of our goal, the lemma below can be thought of as reducing an arbitrary “fibration” to a “fibration” with irreducible fibers.

**Lemma 7.6.** *Suppose  $(C_s)_{s \in S}$  is a definable family of curves on quasi-affine  $V$ . There is a family  $(D_t)_{t \in T}$  of irreducible curves on  $V$  and  $N \in \mathbb{N}$  such that for all  $s \in S$ ,  $C_s$  is a union at most  $N$  irreducible curves from  $(D_t)_{t \in T}$  and  $N$  many points.*

*Proof.* We will give this proof as a demonstration of a standard technique which we will omit details in the later proofs. Suppose  $(C_s)_{s \in S}$  is as given. The idea is to use compactness to show that  $S$  can be definably partitioned into  $S_1, \dots, S_k$  such that for every  $i \in \{1, \dots, k\}$ , the subfamily  $(C_s)_{s \in S_i}$  behaves uniformly in such a way which make the desired conclusion obvious.

For the remaining part of the proof, let  $\mathbb{F}'$  be an elementary extension of  $\mathbb{F}$ . We have the following facts:

- (1)  $(C_s)_{s \in S}(\mathbb{F}')$  is a family of curves on  $V(\mathbb{F}')$  by the preceding lemma;
- (2) every curve over  $\mathbb{F}'$  is a union of some  $k$  irreducible curves and some  $l$  points;
- (3) for every curve  $C'$  on  $\mathbb{F}'$ ; there are systems  $P, Q \in \mathbb{Z}[x, y]$  such that  $C' = Z(P(x, b)) \setminus Z(Q(x, b))$  for  $b' \in (\mathbb{F}')^n$ ; recall that  $y = (y_1, \dots, y_n)$ .

Let  $\mathcal{C}$  be a choice of  $k, l, n \in \mathbb{N}$  and systems  $P_1, \dots, P_k, Q_1, \dots, Q_k$  of polynomials in  $\mathbb{Z}[x, y]$ . We note that there are only countably many such  $\mathcal{C}$ . Let  $R_{\mathcal{C}}$  be the set of  $(s', b') \in S(\mathbb{F}') \times (\mathbb{F}')^n$  such that

- (i) for  $i \in \{1, \dots, k\}$ , the set  $Z(P_i(x, b')) \setminus Z(Q_i(x, b'))$  is an irreducible curve;
- (ii)  $C'_{s'} = \{a^{(1)}, \dots, a^{(l)}\} \cup \bigcup_{i=1}^k Z(P_i(x, b')) \setminus Z(Q_i(x, b'))$  where  $C'_{s'}$  is the curve in in the family  $(C_s)_{s \in S}(\mathbb{F}')$  corresponding to  $s'$  and  $a^{(1)}, \dots, a^{(l)}$  are some  $l$  points on  $V(\mathbb{F}')$ .

Let  $R_{\mathcal{C}}$  be defined likewise with  $\mathbb{F}'$  replaced by  $\mathbb{F}$ . Using lemma 7.4, it is easy to see that  $R_{\mathcal{C}}$  is definable and is moreover equal to  $R_{\mathcal{C}}(\mathbb{F}')$ . Let  $R_{\mathcal{C}}^1$  be the projection of  $R_{\mathcal{C}}$  on  $S$ . Then  $R_{\mathcal{C}}^1(\mathbb{F}')$  is the projection of  $R_{\mathcal{C}}(\mathbb{F}')$  for all elementary extension  $\mathbb{F}'$  of  $\mathbb{F}$ .

We obtain a partition  $S_1, \dots, S_k$  of  $S$  such that for each  $i \in \{1, \dots, k\}$  there is a choice of  $\mathcal{C}$  as in the preceding paragraph such that  $S_k \subseteq R_{\mathcal{C}}^1$ . By (2) and (3), for all  $\mathbb{F}'$  elementary extension of  $\mathbb{F}$ , we have that

$$S(\mathbb{F}') = \bigcup_{\mathcal{C}} R_{\mathcal{C}}^1(\mathbb{F}').$$

By a standard compactness argument and the fact that  $\mathbb{F}'$  was chosen arbitrarily, there are finitely many choices  $\mathcal{C}_1, \dots, \mathcal{C}_k$  obtained in a similar way as  $\mathcal{C}$  such that  $S = \bigcup_k R_{\mathcal{C}_k}^1$ . By routine manipulations, we obtain  $S_1, \dots, S_k$  as desired.

We next construct the family  $(D_t)_{t \in T}$  as describe. We first consider the special case where there is a choice  $\mathcal{C}$  as in the preceding paragraph such that  $S \subseteq R_{\mathcal{C}}^1$ . Choose distinct elements  $b_1, \dots, b_k \in F$ . Let  $T$  be the set of  $t = (t_1, \dots, t_{n+1})$  in  $\mathbb{F}^{n+1}$  such that for some  $i \in \{1, \dots, k\}$ ,  $t_{n+1} = b_i$  and

$$Z(P_i(x, t_1, \dots, t_n)) \setminus Z(Q_i(x, t_1, \dots, t_n))$$

is an irreducible curve. For  $t \in T$ , let

$$D_t = Z(P_i(x, t_1, \dots, t_n)) \setminus Z(Q_i(x, t_1, \dots, t_n)).$$

Let  $N = \max\{k, l\}$  and check that  $(D_t)_{t \in T}$  and  $N$  are as desired. The general case follows easily from the above special case as the disjoint union of finitely many definable families is definable.  $\square$

We next account for the fact that the curves in the “fibration” might not be smooth.

**Lemma 7.7.** *Suppose  $(C_s)_{s \in S}$  is a definable family of irreducible curves on quasi-affine  $V$  and  $f \in \mathbb{F}(V)$ . For each  $s \in S$ , suppose  $D_s$  is a smooth projective curve birationally equivalent to  $C_s$ . Then there is  $N \in \mathbb{N}$  such that as  $s$  ranges over  $S$ , either  $|C_s \cap \text{Dom} f|$  is finite or there is an open subset  $U_s$  of  $C_s$  and an open subset  $W_s$  of  $D_s$  such that  $W_s \subseteq \text{Dom} f$ ,  $U_s$  is isomorphic to  $W_s$  and  $|(C_s \setminus U_s) \cup (D_s \setminus W_s)| < N$ .*

*Proof.* Let  $\mathbb{F}'$  be an elementary extension of  $\mathbb{F}$  and  $\mathbb{P}^m$  be the  $m$ -dimensional projective space over  $\mathbb{F}'$ . We have the following facts in addition to (1), (2) and (3) in the proof of the preceding lemma:

- (1) every irreducible curve over  $\mathbb{F}'$  is birational to a smooth irreducible closed curve on  $\mathbb{P}^3$ ;
- (2) a closed curve on  $\mathbb{P}^3$  is the zero set of a system of homogeneous polynomials which can be obtained by adding parameters from  $\mathbb{F}'$  into a system of polynomials with coefficient in  $\mathbb{Z}$ ;
- (3) if  $C, D$  are quasi-affine curves over  $\mathbb{F}'$ , a rational map from  $C$  to  $D$  is given by substituting parameters from  $\mathbb{F}'$  into a fraction polynomial with coefficient from  $\mathbb{Z}$ ;
- (4) if two curves  $C, D$  over  $\mathbb{F}$  are birational, there is open  $U \subseteq C$  and  $W \subseteq D$  such that  $U, W$  are isomorphic and  $(C \setminus U) \cup (D \setminus W)$  is finite;
- (5) A curve  $C$  is smooth at  $p \in C$  if and only if the zero set of the Jacobian at  $p$  of the system of equations defining  $C$  is one-dimensional.

The proof proceeds in a similar fashion as the preceding lemma.  $\square$

We next address (2) in the remark above lemma 7.4. The following lemma concerns with the component  $2g - 2$  in the right-hand-side expression of Lemma 7.3.

**Lemma 7.8.** *Suppose  $(C_s)_{s \in S}$  is a definable family of irreducible curves on quasi-affine  $V$ . There is  $N \in \mathbb{N}$  such that for all  $s \in S$ ,  $C_s$  has genus  $g_s < N$ .*

*Proof.* Let  $\mathbb{F}'$  be an elementary extension of  $\mathbb{F}$ , we have the following facts:

- (1) every irreducible curve is birational to a closed curve in  $(\mathbb{F}')^2$ ;
- (2) a closed curve on  $(\mathbb{F}')^2$  is the zero set of a system of polynomials which can be obtained by adding parameters from  $\mathbb{F}'$  into a system of polynomials with integer coefficients;
- (3) if  $C, D$  are quasi-affine curves over  $\mathbb{F}'$ , a rational map from  $C$  to  $D$  is given by substituting parameters from  $\mathbb{F}'$  into a rational function with integer coefficients;
- (4) the geometric genus is a birational invariants of irreducible curves;
- (5) the geometric genus of a curve in  $(\mathbb{F}')^2$  is bounded above by its arithmetic genus
- (6) every irreducible curve in  $F^2$  is the zero set of an irreducible polynomial  $P \in F[z_1, z_2]$ ;
- (7) the arithmetic genus of the zero set of irreducible  $P \in F[z_1, z_2]$  is

$$\frac{1}{2} \deg P (\deg P - 1).$$

Again, the proof is similar to Lemma 7.6 with the use Lemma 7.4. Alternatively, this lemma may be proven using flattening straightification and semi-continuity theorems.  $\square$

Suppose  $C$  is an irreducible curve on  $V$  and  $f \in \mathbb{F}(V)$  is such that  $\text{Dom}(f) \cap C$  is open in  $C$ . Then  $f|_C$  is in  $\mathbb{F}(C)$ . Lemma 7.2 shows that if  $C, f$  are moreover definable over  $\mathbb{F}_q$ , then  $f|_C$  is in  $\mathbb{F}_q(C)$ . The following lemma allow us to deal with the remaining part of the right-hand-side expression of Lemma 7.3

**Lemma 7.9.** *Suppose  $(C_s)_{s \in S}$  is a definable family of irreducible curves on a quasi-affine variety  $V$ ,  $f$  is in  $\mathbb{F}(V)$ . There is  $N \in \mathbb{N}$  such that for all  $\mathbb{F}_q$  and all  $C_s$  in the above family with  $V, C_s, f$  definable over  $\mathbb{F}_q$ ,  $\text{Dom}(f) \cap C_s$  open in  $C_s$  and  $f|_{C_s}$  non-constant on  $C_s$  we have:*

$$\left| \sum_{v \in \tilde{Z}_{C_s, f_s}(\mathbb{F}_q) \cup \tilde{P}_{C_s, f_s}(\mathbb{F}_q)} \deg(v) \right| < N \text{ where } f_s = f|_{C_s}.$$

*Proof.* Suppose  $(C_s)_{s \in S}, f$  are as in the first statement of the lemma and  $\mathbb{F}_q, C_s, f_s$  are as in the second statement of the lemma. By definition,  $v \in \tilde{Z}_{C_s, f_s}$  implies  $v(f_s) > 0$ . Hence,

$$\left| \sum_{v \in \tilde{Z}_{C_s, f_s}(\mathbb{F}_q)} \deg(v) \right| \leq \left| \sum_{v \in \tilde{Z}_{C_s, f_s}(\mathbb{F}_q)} v(f) \deg(v) \right| \leq [\mathbb{F}_q(C_s) : \mathbb{F}_q(f|_{C_s})]$$

where the second inequality is by [Sti09, Prop 1.3.3]. As  $C_s$  is absolutely irreducible, we have  $[\mathbb{F}_q(C_s) : \mathbb{F}_q(f|_{C_s})] = [\mathbb{F}(C_s) : \mathbb{F}(f|_{C_s})]$  [Per91, (1.2)]. We also have that

$$[\mathbb{F}(C_s) : \mathbb{F}(f)] = \max_{a \in \text{Im } f} |f^{-1}(a) \cap C_s|.$$

By algebraic boundedness of ACF or an argument similar to Lemma 7.6, the above has an upper bound  $N_1 \in \mathbb{N}$  independent of the choice of  $C_s$  satisfying the stated properties. Note that  $v \in \tilde{P}_{C_s, f}$  if and only if  $v \in \tilde{Z}_{C_s, 1/f}$ . Therefore, the above argument also gives us an upper bound  $N_2 \in \mathbb{N}$  of  $|\sum_{v \in \tilde{P}_{C_s, f}(\mathbb{F}_q)} \deg(v)|$  independent of the choice of  $C_s$  satisfying the stated properties. Clearly,  $N = N_1 + N_2$  is the desired upper bound.  $\square$

**Lemma 7.10.** *Suppose  $(C_s)_{s \in S}$  is a definable family of curve on a quasi-affine variety  $V$  and  $f$  is in  $\mathbb{F}(V)$ . Then there is  $N \in \mathbb{N}$  such that for all  $\mathbb{F}_q$  and all  $C_s$  in the above family with  $V, f$  definable over  $\mathbb{F}_q$ ,  $\text{Dom}(f) \cap C_s$  open in  $C_s$  and  $f$  non-constant on  $C_s$ , we have:*

$$\left| \sum_{a \in \text{Dom } f(\mathbb{F}_q) \cap C_s} \chi(f(a)) \right| \leq Nq^{\frac{1}{2}}.$$

*Proof.* Suppose  $(C_s)_{s \in S}, f$  are as stated. By Lemma 7.4, we can arrange that  $(C_s)_{s \in S}$  is a definable family of irreducible curves.

We first show that there is  $N_1 \in \mathbb{N}$  such that for all  $\mathbb{F}_q$  and all  $C_s$  as in the statement of the lemma and  $C_s$  is moreover not definable over  $\mathbb{F}_q$  then

$$|\text{Dom } f(\mathbb{F}_q) \cap C_s| < N_1.$$

Suppose  $\mathbb{F}_q$  and  $C_s$  are as stated. Let  $\text{Frob}$  denote the map  $\mathbb{F} \rightarrow \mathbb{F}, a \rightarrow a^q$  and the induced map on  $\mathbb{F}^m$  for  $m > 0$ . Then  $\text{Dom } f(\mathbb{F}_q) \cap C_s \subseteq C_s \cap \text{Frob}^{-1}(C_s)$  which is finite. The conclusion follows from the preceding lemma noting that the family  $(\text{Frob}^{-1}C_s)_{s \in S}$  is also definable.

We next show that there is  $N_2 \in \mathbb{N}$  such that for all  $\mathbb{F}_q$  and all  $C_s$  as in the statement of the lemma and  $C_s$  is moreover definable over  $\mathbb{F}_q$  then

$$\left| \sum_{a \in \text{Dom} f(\mathbb{F}_q) \cap C_s} \chi(f(a)) \right| \leq N_2 q^{\frac{1}{2}}.$$

Suppose  $\mathbb{F}_q$  and  $C_s$  are as stated. Then  $f_s = f|_{C_s}$  is a non-constant element in  $\mathbb{F}_q(C_s)$ . It immediately follows that  $\text{Dom} f(\mathbb{F}_q) \cap C_s = \text{Dom} f_s(\mathbb{F}_q)$ . We note that the normalization of  $C_s$  is also definable over  $\mathbb{F}_q$ . Therefore, for each  $s \in S$ , there is a smooth projective curve  $D_s$  definable over  $\mathbb{F}_q$  and birational equivalence  $\iota : C_s \rightarrow D_s$ . Let  $h_s$  be the image of  $f_s$  under the isomorphism from  $F(C_s)$  to  $F(D_s)$  induced by  $\iota$ . Applying Lemma 7.3 noting that the right-hand-side expression of this lemma is invariant under birational equivalence, we get:

$$\left| \sum_{a \in \text{Dom} h_s(\mathbb{F}_q)} \chi(h_s(a)) \right| \leq \left( 2g_s - 2 + \sum_{v \in \tilde{Z}_{C_s, f_s}(\mathbb{F}_q) \cup \tilde{P}_{C_s, f_s}(\mathbb{F}_q)} \deg(v) \right) \sqrt{q}.$$

Let  $N_3$  be the bound in 7.8 and  $N_4$  be the bound in 7.9. By Lemma 7.7, there is  $U_s$  open in  $C_s$ ,  $W_s$  open in  $D_s$  such that the restriction of  $\iota$  is an isomorphism from  $U_s$  to  $W_s$  and  $|C_s \setminus U_s| + |D_s \setminus W_s| < N_5$  where  $N_5$  is the bound in Lemma 7.7. Putting everything together we have:

$$\left| \sum_{a \in \text{Dom} f(\mathbb{F}_q) \cap C_s} \chi(f(a)) \right| \leq \left| \sum_{a' \in \text{Dom} f(\mathbb{F}_q) \cap C'_s} \chi(f(a')) \right| + N_5 \leq (2N_3 + N_4)q^{1/2} + N_5.$$

Then  $N_2 = 2N_3 + N_3 + N_5$  is the desired bound which is independent of the choice of  $C_s$  with the stated properties.

Finally, it is easy to see that  $N = N_1 + N_2$  where  $N_1, N_2$  are obtained in the previous paragraphs is a desired bound for the lemma.  $\square$

**Proposition 7.11.** *Suppose  $V \subseteq \mathbb{F}^m$  is a quasi-affine of dimension  $d$  and  $f \in F(V)$  is nonconstant on  $V$ . There is  $N \in \mathbb{N}$  such that if  $V, f$  are definable over  $\mathbb{F}_q$ , then:*

$$\left| \sum_{a \in \text{Dom} f(\mathbb{F}_q)} \chi(f(a)) \right| \leq N q^{d-\frac{1}{2}}.$$

*Proof.* Suppose  $V, d$  and  $f$  are as given. We make a number of observations and arrangements. As  $f$  is non-constant,  $d > 0$ . If  $V, f$  are definable over  $\mathbb{F}_q$ , then  $\text{Dom} f$  is also definable over  $\mathbb{F}_q$ . We can therefore replace  $V$  with  $\text{Dom} f$  and assume that  $f$  is regular on  $V$ . Replacing  $V$  with the graph of  $f$  and replace  $m$  with  $m + 1$  if necessary, we arrange that  $f = \pi_m$  where  $\pi_m : \mathbb{F}^m \rightarrow \mathbb{F}$  is the projection to the first coordinate for  $m > 0$ .

We will show by induction on dimension an auxiliary result which implies that  $V$  has a “good fibration”. Let  $F$  be a finite subfield of  $\mathbb{F}$  such that  $V$  is definable over  $F$  and  $F$  is minimal with respect to these properties. For  $m > 0$ , let  $\rho_m : \mathbb{F}^m \rightarrow \mathbb{F}^{m-1}$  be the projection on the last  $m - 1$  coordinates. We will construct a (Zariski) open subset  $U$  of  $V$ , an open subset  $D$  of  $\mathbb{F}^d$ , an open subset  $S$  of  $\mathbb{F}^{d-1}$  and a “reduction map”  $r : V \rightarrow \mathbb{F}^d$  with the following properties:

- (1)  $U, D, S, r$  are definable over  $F$ ;
- (2)  $U \subseteq \text{Dom}(r)$ ,  $r(U) = D$  and  $\rho_d(D) = S$ ;
- (3)  $\pi_m = \pi_d \circ r$  on  $U$ .

We consider the special case when  $m = d$ . Then  $V$  is open in  $\mathbb{F}^m = \mathbb{F}^d$  and the image of  $V$  under  $\rho$  contains an open subset  $S$  of  $\mathbb{F}^{d-1}$ . We can arrange that  $S$  satisfies (2) of Lemma 7.2 and so  $F$ -definable. Let  $U = D = h^{-1}(S) \cap V$  and  $r$  be the identity map. We check that this choice satisfies the desired conditions.

We consider another special case where  $d = 1$ . As  $\pi_m$  is non-constant, by an argument similar to the preceding paragraph, there is an open set  $D$  of  $\mathbb{F}$  such that  $D$  is definable over  $F$  and  $D$  is contained in the image of  $\pi_m$ . Let  $S$  be the set of one element  $\mathbb{F}^0$ ,  $U = V \cap \pi_m^{-1}(D)$  and  $r$  be  $f \upharpoonright U$ . We check that this choice satisfies the desired conditions.

Towards the use of induction, suppose  $m > d$ , and  $d > 1$  and we have proven the statement for all  $V', m'$  and  $d'$  with similar settings such that  $m' < m$ . Let  $\tau_m : \mathbb{F}^m \rightarrow \mathbb{F}^{m-1}$  be the projection on the first  $m-1$  coordinates. Using Lemma 7.2 and arguing similarly as the third paragraph to obtain an open subset  $V'$  of  $\tau_m(V)$  such that  $V'$  is definable over  $F$ . By induction hypothesis, we can choose  $U', D', S', r'$  satisfies the desired condition for  $V'$ ,  $m-1$  and  $d' = \dim(V')$ . Consider the case where  $d' = d$ . Set

$$U = \tau_m^{-1}(U') \cap V, \quad D = D', \quad S = S' \text{ and } r = r' \circ \tau_m.$$

We check that this satisfies the desired condition. Consider the case where  $d' = d-1$ . Set

$$U = \tau_m^{-1}(U') \cap V, \quad D = \tau_d^{-1}(D'), \quad S = \tau_{d-1}^{-1}(S');$$

$$r : V \rightarrow F^d, \quad a = (a_1, \dots, a_m) \mapsto (r' \circ \tau_m(a), a_m),$$

Shrink  $U, D, S$  further if needed we make  $r(U) = D$ ,  $\rho_d(D) = S$  and  $U, D$  definable over  $F$ . We can check that all the conditions are satisfied.

Suppose  $V$  is definable over  $\mathbb{F}_q$ . We claim that  $F \subseteq \mathbb{F}_q$ . Let  $\sigma$  be in  $\text{Gal}(\mathbb{F} | \mathbb{F}_q)$ . Then as  $\text{Gal}(\mathbb{F} | \mathbb{F}_p)$  is abelian,  $\langle \text{Gal}(F | \mathbb{F}_p), \sigma \rangle$  is  $\text{Gal}(\mathbb{F} | F')$  where  $F' \subseteq F$  and  $F'$  in an extension of  $\mathbb{F}_p$ . Then every elements of  $\text{Gal}(\mathbb{F} | F')$  fixes  $V$  set-wise and so  $V$  is definable over  $F'$ . By minimal assumption of  $F$ , we must have  $F' = F$ . Therefore  $\sigma$  is in  $\text{Gal}(\mathbb{F} | F)$ . The desired conclusion follows.

Therefore,  $U, D, S, r$  obtained in the previous paragraphs are also definable over  $\mathbb{F}_q$ . For each  $s \in S$  set

$$L_s = D \cap \rho_d^{-1}(s) \text{ and } C_s = U \cap r^{-1}(L_s) = U \cap (\rho_d \circ r)^{-1}(s).$$

As  $r$  is  $\mathbb{F}_q$ -definable, by Lemma 7.2, if  $a \in U(\mathbb{F}_q)$ , then  $\rho_d \circ r(a)$  is in  $\mathbb{F}_q^{d-1}$ . Therefore,  $U(\mathbb{F}_q) = \bigcup_{s \in S(\mathbb{F}_q)} C_s(\mathbb{F}_q)$ . For each  $s \in S(\mathbb{F}_q)$ , we also have  $C_s$  is definable over  $\mathbb{F}_q$  and  $\pi_m(C_s) = \pi_d \circ r(C_s) = \pi_d(L_s)$  is nonconstant as  $L_s$  is open in  $\pi_d^{-1}(s)$ . Hence,

$$\left| \sum_{a \in U(\mathbb{F}_q)} \chi(f(a)) \right| \leq \sum_{s \in S(\mathbb{F}_q)} \left| \sum_{a \in C_s(\mathbb{F}_q)} \chi(f(a)) \right| \leq q^{d-1} B_1 q^{\frac{1}{2}} = B_1 q^{d-\frac{1}{2}}$$

with  $B_1$  the bound from Lemma 7.10. On the other hand,

$$\left| \sum_{a \in (V \setminus U)(\mathbb{F}_q)} \chi(f(a)) \right| \leq |(V \setminus U)(\mathbb{F}_q)| \leq B_2 q^{d-1}$$

with  $B_2$  the bound given by Lemma 2.1. Thus,  $B = B_1 + B_2$  is the desired bound.  $\square$

## REFERENCES

- [Ax68] James Ax, *The elementary theory of finite fields*, Ann. of Math. (2) **88** (1968), 239–271.
- [BG06] Enrico Bombieri and Walter Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs, vol. 4, Cambridge University Press, Cambridge, 2006.
- [Cha97] Zoé Chatzidakis, *Model theory of finite fields and pseudo-finite fields*, Ann. Pure Appl. Logic **88** (1997), no. 2-3, 95–108.
- [Che14] Artem Chernikov, *Theories without the tree property of the second kind*, Ann. Pure Appl. Logic **165** (2014), no. 2, 695–723.
- [CK90] C. C. Chang and H. J. Keisler, *Model theory*, third ed., Studies in Logic and the Foundations of Mathematics, vol. 73, North-Holland Publishing Co., Amsterdam, 1990.
- [Coh93] Henri Cohen, *A course in computational algebraic number theory*, Graduate Texts in Mathematics, vol. 138, Springer-Verlag, Berlin, 1993.
- [CP98] Z. Chatzidakis and A. Pillay, *Generic structures and simple theories*, Ann. Pure Appl. Logic **95** (1998), no. 1-3, 71–92.
- [Del77] Pierre Deligne, *Applications de la formule des traces aux sommes trigonométriques*, pp. 168–232, Springer Berlin Heidelberg, Berlin, Heidelberg, 1977.
- [Del80] ———, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 52, 137–252.
- [Gün08] Ayhan Güneydin, *Model theory of fields with multiplicative subgroups*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2008.
- [Joh16] William Johnson, *Fun with fields*, Ph.D. thesis, University of California, Berkeley, 2016.
- [Kow07] E. Kowalski, *Exponential sums over definable subsets of finite fields*, Israel J. Math. **160** (2007), 219–251.
- [LW54] Serge Lang and André Weil, *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827.
- [Mar02] David Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002, An introduction.
- [Per91] Marc Perret, *Multiplicative character sums and Kummer coverings*, Acta Arith. **59** (1991), no. 3, 279–290.
- [RZ60] Abraham Robinson and Elias Zakon, *Elementary properties of ordered abelian groups*, Trans. Amer. Math. Soc. **96** (1960), 222–236.
- [SS03] Elias M. Stein and Rami Shakarchi, *Fourier analysis*, Princeton Lectures in Analysis, vol. 1, Princeton University Press, Princeton, NJ, 2003, An introduction.
- [SS12] Saharon Shelah and Pierre Simon, *Adding linear orders*, J. Symbolic Logic **77** (2012), no. 2, 717–725.
- [Sti09] Henning Stichtenoth, *Algebraic function fields and codes*, second ed., Graduate Texts in Mathematics, vol. 254, Springer-Verlag, Berlin, 2009.
- [Tow13] Brett Townsend, *Dimension theory in dense regular groups*, Ph.D. thesis, Wesleyan University, 2013.
- [Tsu01] Akito Tsuboi, *Random amalgamation of simple theories*, MLQ Math. Log. Q. **47** (2001), no. 1, 45–50.
- [vdDGn06] Lou van den Dries and Ayhan Güneydin, *The fields of real and complex numbers with a small multiplicative group*, Proc. London Math. Soc. (3) **93** (2006), no. 1, 43–81.
- [Wei81] Volker Weispfenning, *Elimination of quantifiers for certain ordered and lattice-ordered abelian groups*, Proceedings of the Model Theory Meeting, vol. 33, 1981, pp. 131–155.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA- CHAMPAIGN, URBANA,  
IL 61801, U.S.A

*E-mail address:* mctran2@illinois.edu